The cross product revisited

In our earlier article on determinant, we explicitly saw that the area of parallelogram spanned by two two-dimensional vectors was given by, up to sign, the determinant of 2×2 matrix composed by these two vectors. We also remarked that the volume of the parallelepiped spanned by n n-dimensional vectors must be given by, up to sign, the determinant of $n \times n$ matrix composed by these n vectors.

Now, let's consider the case when $n = 3$. See Fig. 1. The volume of this parallelepiped is given by:

$$V = \begin{vmatrix} d_x & d_y & d_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1}$$

where

$$\begin{aligned} \vec{d} &= d_x i + d_y j + d_z k \\ \vec{a} &= a_x i + a_y j + a_z k \\ \vec{b} &= b_x i + b_y j + b_z k \end{aligned}$$

Now, let's say that θ is the angle between $\vec{a}$ and $\vec{b}$, and α is the angle between $\vec{c}$ and $\vec{d}$ where $\vec{c}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$, and with magnitude given by $|\vec{a}||\vec{b}|\sin\theta$. Actually, there are two vectors that satisfy this property, since there are two directions perpendicular to two given directions. Of course, these two directions are directly opposite to each other. For the time being, we won't bother with this fact.

Then, as the area of base of the parallelepiped is $|\vec{a}||\vec{b}|\sin\theta$, and the height of the parallelepiped is $h = |\vec{d}|\cos\alpha$ the volume of the parallelepiped is given by following:

$$V = (|\vec{a}||\vec{b}|\sin\theta)|\vec{d}|\cos\alpha = \vec{c}\cdot\vec{d} \tag{2}$$

Now, to express $\vec{c}$ in components, let's do some calculations as follows:

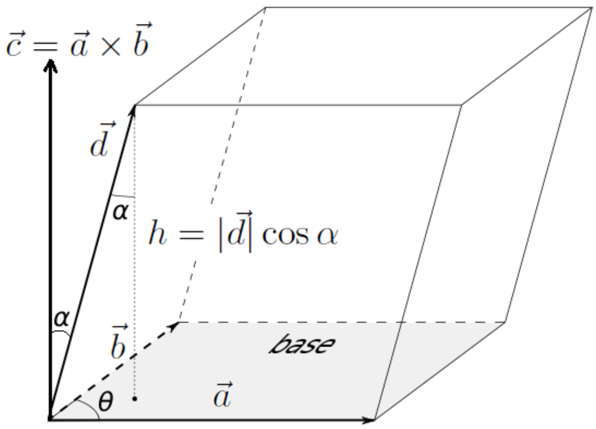

Figure 1: parallelepiped

$$
\begin{aligned}
\begin{vmatrix} d_x & d_y & d_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} &= d_x \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - d_y \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + d_z \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \\
&= \left(i \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - j \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + k \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \right) \cdot (d_x i + d_y j + d_z k) \\
&= \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \cdot \vec{d} \qquad (3)
\end{aligned}
$$

So, we conclude:

$$\vec{c} = i \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - j \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + k \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad (4)$$

Now, we see that $\vec{c}$ is exactly the "cross product" of $\vec{a}$ and $\vec{b}$ as follows:

$$
\begin{aligned}
\vec{a} \times \vec{b} &= (a_x i + a_y j + a_z k) \times (b_x i + b_y j + b_z k) \\
&= (a_y b_z - a_z b_y) i + (a_z b_x - a_x b_z) j + (a_x b_y - a_y b_x) k \\
&= i \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - j \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + k \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}
\end{aligned}
$$

Thus, we can better understand why the cross product is defined in such a way it is; earlier in our article "cross product," it was merely seen as a coincidence that the cross product satisfies various properties. Now, we have the reason behind.

Problem 1. Show that the volume of the parallelepiped spanned by $\vec{a}$,$\vec{b}$, and $\vec{d}$ can be expressed using the cross product as follows (Hint[1]):

$$|\vec{d}\cdot(\vec{a}\times\vec{b})| = V \tag{5}$$

Problem 2. From the properties of determinant, explain why the followings must hold.

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a} \tag{6}$$
$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c} \tag{7}$$
$$(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c} \tag{8}$$
$$\vec{a}\times\vec{a} = 0 \tag{9}$$
$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = -\vec{b}\cdot(\vec{a}\times\vec{c}) \tag{10}$$

Problem 3. Using your answer to Problem 4 in our earlier article "Levi-Civita symbol, " prove the following:

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \tag{11}$$

Problem 4. Let, $\vec{L}_1+\vec{L}_2+\vec{L}_3+\vec{L}_4 = 0$. Then, show that the volume of parallelepiped spanned by any arbitrarily chosen three vectors among $\vec{L}_1,\vec{L}_2,\vec{L}_3,\vec{L}_4$ doesn't depend on which three vectors we choose from the four vectors. This fact is applied when calculating the eigenvalues of volume operator in loop quantum gravity.

Summary

- $$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

- The volume of the parallelepiped spanned by $\vec{a}$, $\vec{b}$ and $\vec{d}$ is given by $\vec{d}\cdot(\vec{a}\times\vec{b})$.

[1] Use (2)