De Rham cohomology

In an earlier article, I introduced differential forms. In this article, I will explain how it is related to topology.

Let me introduce closed forms and exact forms. A closed form w is a form such that

$$dw = 0 \tag{1}$$

An exact form w is a form such that

$$w = df \tag{2}$$

for some f.

As $d^2 = 0$, we have

$$d^2 f = d(df) = dw = 0 \tag{3}$$

This result implies that an exact form is always a closed form but not vice versa. De Rham cohomology tells us how many more closed forms there are than exact forms. The number depends on the topology of the background where these differential forms are defined.

To understand this, let's consider a torus. See Fig. 1.

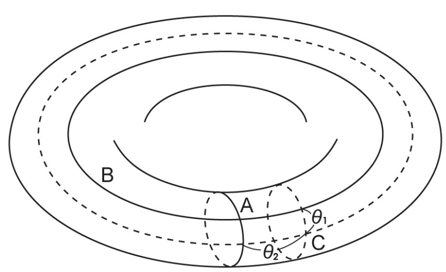

Figure 1: Coordinates θ_1 and θ_2

As a torus is two-dimensional, we can assign two periodic coordinates to specify any point on it (When we say a torus in mathematics, we mean the surface of the torus, not the interior which is called a "toroid." Therefore, it's two-dimensional rather than three-dimensional). Let's denote these coordinates as θ_1 and θ_2 and say that each of them ranges from 0 to 2π. Of course the point θ_1 is the same point as $\theta_1 + 2\pi$, and similarly for θ_2. Given this, what would be the one-forms on this torus? Naturally, we can write a one-form w as following:

$$w = f(\theta_1, \theta_2)d\theta_1 + g(\theta_1, \theta_2)d\theta_2 \tag{4}$$

In this case, there are two bases for the one-form: $d\theta_1$ and $d\theta_2$.

These one-forms are closed. Now let's see whether they are exact. If they are exact, we can write:

$$d\theta_1 = d(\alpha_1) \tag{5}$$

$$d\theta_2 = d(\alpha_2) \tag{6}$$

for some α_1 and α_2. Notice that $d\theta_1$ doesn't necessarily mean $d(\theta_1)$, even though this notation may be confusing. This confusion should be cleared up as you read further into this article.

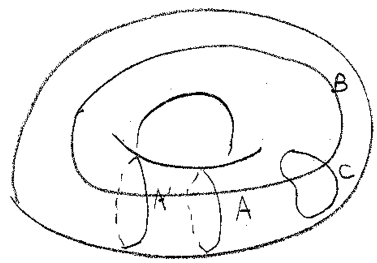

Figure 2: A, A', B, C

Now, let's integrate each of the one-forms on the circles A and B which are shown in Fig. 2, assuming they are exact, and see what happens. Using Stokes' theorem, we can write:

$$\int_0^{2\pi} d\theta_1 = \int_A d\theta_1 = \int_A d(\alpha_1) = \int_{\partial A} \alpha_1 = 0 \tag{7}$$

where we have used the fact that a circle has no boundary.

Similarly, we can write

$$\int_0^{2\pi} d\theta_2 = \int_B d\theta_2 = \int_B d(\alpha_2) = \int_{\partial B} \alpha_2 = 0 \tag{8}$$

However, the above integrals should not be zero as

$$\int_0^{2\pi} d\theta_1 = \int_0^{2\pi} d\theta_2 = 2\pi \tag{9}$$

Therefore, $d\theta_1$ and $d\theta_2$ are examples of one-forms that are closed but not exact. Notice that they are so because the coordinates θ_1 and θ_2 are periodic. This periodicity is unavoidable if we try to define coordinates on a torus. Also, notice that this unavoidability depends on the topology of the background on which the differential forms are defined.

We call $d\theta_1$ and $d\theta_2$ the de Rham cohomology elements of $H^1_{dR}(\text{Torus})$, where the "1" denotes a one-form and the "dR" denotes "de Rham." Cohomology counts the number of forms that are closed but not exact under an operator Q that satisfies $Q^2 = 0$, and de Rham cohomology is given by the case where Q is an exterior derivative d. As there are two linearly independent elements in $H^1_{dR}(\text{Torus})$, we say $H^1_{dR}(\text{Torus}) = \mathbb{R}^2$ where the "2" here denotes the number of linearly independent elements. In other words, the vector space spanned by $d\theta_1$ and $d\theta_2$ have two dimensions.

What would $H^0_{dR}(\text{Torus})$ be? A constant function c satisfies $dc = 0$ which means that it's closed, but it's not exact since there is no object b that would satisfy $db = c$ since b must be a "-1 form" and there is no such thing as a "-1 form." Therefore, a constant function is the basis of the de Rham cohomology of 0-forms and is the only basis. So,

$$H^0_{dR}(\text{Torus}) = \mathbb{R}^1 \tag{10}$$

Then, what would $H^2_{dR}(\text{Torus})$ be? Consider the fact that naturally any two-forms must be expressed in terms of the wedge product of two one-forms. Therefore, we can denote a two form β as:

$$\beta = k(\theta_1, \theta_2) d\theta_1 \wedge d\theta_2 \tag{11}$$

So the basis of the two-form on a torus is $d\theta_1 \wedge d\theta_2$. It is easy to see that it is closed.

$$d(d\theta_1 \wedge d\theta_2) = dd\theta_1 \wedge d\theta_2 - d\theta_1 \wedge dd\theta_2 = 0 \wedge d\theta_2 - d\theta_1 \wedge 0 = 0 \tag{12}$$

However, it is not exact, since neither $d\theta_1$ nor $d\theta_2$ is exact. (If they were, we would be able to express $d\theta_1 \wedge d\theta_2$ as $d(\theta_1 \wedge d\theta_2)$ or $d(-d\theta_1 \wedge \theta_2)$. Therefore, $H^2_{dR}(\text{Torus})$ is $\mathbb{R}^1$.

Now, what would be the de Rham cohomology elements of $\mathbb{R}^n$? ($\mathbb{R}^n$ is the usual n-dimensional Euclidean space.)

$H^0_{dR}(\mathbb{R}^n) = \mathbb{R}^1$ from the same reason as $H^0_{dR}(\text{Torus}) = \mathbb{R}^1$; a constant function is the basis of $H^0_{dR}(\mathbb{R}^n) = \mathbb{R}^1$. However, $H^k_{dR}(\mathbb{R}^n) = \mathbb{R}^0$ for k other than 0 since the natural variables specifying the position of R^n can always be assigned non-periodically.

Now let's calculate the de Rham cohomology of one more complicated example. As a first step to this end, notice that a torus is the direct product of two circles. In other words, a point on a torus is specified by its location in the first circle A and its location in the second circle B. Recall that the coordinates were θ_1 and θ_2. We express this fact as

$$\text{torus} = S^1 \times S^1 \tag{13}$$

where S^1 denotes the circle. Similarly, we can define a 4-dimensional torus T^4 by $S^1 \times S^1 \times S^1 \times S^1$.

Now let's calculate the de Rham cohomology of this manifold. First of all, $H^0_{dR}(T^4) = \mathbb{R}^1$ from the same reason as before: its element is a constant function. Second, $H^1_{dR}(T^4) = \mathbb{R}^4$. The four elements are $d\theta_1, d\theta_2, d\theta_3$, and $d\theta_4$. Third, $H^2_{dR}(T^4) = \mathbb{R}^6$. The six elements are $d\theta_1 \wedge d\theta_2, d\theta_1 \wedge d\theta_3, d\theta_1 \wedge d\theta_4, d\theta_2 \wedge d\theta_3, d\theta_2^d\theta_4$, and $d\theta_3 \wedge d\theta_4$. Fourth, $H^3_{dR}(T^4) = \mathbb{R}^4$. The four elements are $d\theta_1 \wedge d\theta_2 \wedge d\theta_3, d\theta_2 \wedge d\theta_3 \wedge d\theta_4, d\theta_1 \wedge d\theta_3 \wedge d\theta_4, d\theta_1 \wedge d\theta_2 \wedge d\theta_3$. Finally, $H^4_{dR}(T^4) = \mathbb{R}^1$. The only element is $d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \wedge d\theta_4$.

Now, let me explain what the Künneth formula is. If an n-dimensional manifold M for which we want to calculate the de Rham cohomology is the direct product of two smaller dimensional manifolds, say k-dimensional manifold X and $(n-k)$ dimensional manifold Y, the de Rham cohomology of M is given by the following formula:

$$H^a_{dR}(M) = \sum_{a=b+c} H^b_{dR}(X) \times H^c_{dR}(Y) \tag{14}$$

Let's see why this should be the case. Let's say $a=3$. Then, we can easily see that a 3-form on M can be expressed as the sum of wedge products consisting of a 0-form in X with a 3-form in Y, a 1-form in X with a 2-form in Y, a 2-form in X with a 1-form in Y, or a 3-form in X and a 0-form in Y. This is not a rigorous proof because we need to know exactly why cohomology elements not expressible as the wedge product of a differential form of X and a differential form of Y don't exist. If you read *Superstring theory, Vol 2* by Green, Schwarz, and Witten, you will learn the reason why. You need to understand harmonic forms, which are explained there. In any case, the Künneth formula was rigorously proved by Künneth in his doctoral thesis in 1922.

Problem 1. Calculate the following:

$$\int_B d\theta_1 = ?, \quad \int_C d\theta_1 = ?, \quad \int_{A'} d\theta_1 = ?, \quad \int_A d\theta_2 = ?, \quad \int_C d\theta_2 = ?$$

As we have gained some intuitive understanding of de Rham cohomology, let me give you its mathematically rigorous definition.

Let M be a differentiable manifold. Then, the rth de Rham cohomology group $H^r(M)$ is an equivalence class defined by

$$\omega \sim \omega + d\psi \tag{15}$$

where ω is a closed r-form and ψ is an $(r-1)$-form.

In our later article "The duality between de Rham cohomology and homology" we will show that H^1_{dR}(Torus)$= \mathbb{R}^2$ is indeed true following from the above definition.

Final comment. After learning de Rham cohomology group, I asked a string theorist, whether de Rham cohomology is used to describe the complicated geometry in string theory such as in extra dimensions. He said that it was, but also commented that cohomology has other applications than in geometry. As I mentioned earlier in this article, whenever an operator Q satisfies $Q^2 = 0$, we can define cohomology. As this condition implies that Q-exact form is always Q-closed form, we can define the equivalence relation $\omega \sim \omega + Q\psi$.

If I use the language of mathematicians, a cohomology group is given by the quotient space ker Q/Im Q. Quotient space "/" means that you reduce the original space by an equivalence relation. ker Q, called the "kernel of Q," is a vector space V that satisfies $Qv = 0$ for $v \in V$ (i.e. v is an element of V), while Im Q, called the "image of Q," is a vector space given by $Q(V)$.

Summary

- A closed form w is a form such that $dw = 0$.

- An exact form w is a form such that $w = df$ for some f.
- $d^2 = 0$ implies, an exact form is always a closed form.
- De Rham cohomology tells us how many more closed forms there are than exact forms.
- The rth de Rham cohomology group $H^r(M)$ is an equivalence class defined by

$$\omega \sim \omega + d\psi$$

  where ω is a closed r-form and ψ is an $(r-1)$-form.