The duality between de Rham cohomology and homology

In this article, we will show that the rth homology group $H_r(M)$ and the rth de Rham cohomology group $H^r(M)$ form a dual space. Let $c \in H_r(M)$, and $\omega \in H^r(M)$. Then, the dual map (i.e., a dot product between c and ω) is given by

$$\langle c|\omega\rangle = \int_c \omega \tag{1}$$

It is easy to check that this map satisfies linearity. i.e.,

$$\langle c|\omega_1 + \omega_2\rangle = \int_c (\omega_1 + \omega_2) = \int_c \omega_1 + \int_c \omega_2 = \langle c|\omega_1\rangle + \langle c|\omega_2\rangle \tag{2}$$

$$\langle c_1 + c_2|\omega\rangle = \int_{c_1+c_2} \omega = \int_{c_1} \omega + \int_{c_2} \omega = \langle c_1|\omega\rangle + \langle c_2|\omega\rangle \tag{3}$$

We also need to check that this linear map is well-defined. i.e., that it doesn't depend on which closed forms you choose to evaluate (1) if they are in the same equivalence class (i.e., if they differ by an exact form). For example, let's say $\omega' = \omega + d\psi$. Then,

$$\langle c|\omega'\rangle = \int_c \omega + \int_c d\psi = \int_c \omega + \int_{\partial c} \psi = \int_c \omega = \langle c|\omega\rangle \tag{4}$$

where we used Stoke's theorem, and the fact that $\partial c = 0$ (i.e., c is a cycle.)

Similarly, we need to check that the linear map doesn't depend on which cycles you choose to evaluate if they are in the same equivalence class (i.e., if they differ by a boundary). For example, let's say $c' = c + \partial d$.

$$\langle c'|\omega\rangle = \int_c \omega + \int_{\partial d} \omega = \int_c \omega + \int_d d\omega = \int_c \omega = \langle c|\omega\rangle \tag{5}$$

where we used Stoke's theorem, and $d\omega = 0$ (i.e., ω is a closed form).

So, the linear map (1) is well-defined. Thus, de-Rham cohomology group $H^r(M)$ is dual to homology group $H_r(M)$; their dimension is the same.

I also want to mention that the exterior derivative is the "transpose" of boundary operator and vice versa. To see this, note that Stoke's theorem says

$$\langle \partial c|\omega\rangle = \langle c|d\omega\rangle = \langle c|d|\omega\rangle \tag{6}$$

Thus, $\langle \partial c| = \langle c|d$ which implies

$$|\partial c\rangle = |d^T c\rangle \tag{7}$$

In complex vector space, it would have been the Hermitian conjugate instead of the transpose, but here we are dealing with real vector space.

We will now show that H^1_{dR}(Torus)$= \mathbb{R}^2$. See Fig. 1 in "De Rham cohomology." An one-form on a torus can be written as

$$\omega = f(\theta_1, \theta_2)d\theta_1 + g(\theta_1, \theta_2)d\theta_2 \tag{8}$$

As a cohomology element, we consider a closed one-form, i.e., $d\omega = 0$, which gives a certain relation between f and g, but its explicit form is not important in our argument.

Anyhow, consider the following integration:

$$\int_{A(\theta_2)} \omega = a_1(\theta_2) \tag{9}$$

where $A(\theta_2)$ is an A-cycle located at θ_2, such as the small dotted circle that passes C in the figure. (**Problem 1.** Show that $a_1(\theta_2)$ doesn't depend on θ_2 if $d\omega = 0$.)

Thus, we can choose any θ_2 to calculate (9). Therefore, we can just write

$$\int_A \omega = a_1 \tag{10}$$

where we set $a_1 = a_1(\theta_2)$ as it doesn't depend on θ_2, and choose an arbitrary θ_2 and express $A = A(\theta_2)$.

Problem 2. Show that the above integration doesn't depend on which ω you choose, if they are in the same equivalence class.

So, once you choose $f(\theta_1, \theta_2)$, then a_1 is determined. They are related by

$$\int_0^{2\pi} f(\theta_1, \theta_2)d\theta_1 = a_1 \tag{11}$$

Now, let me ask you a question. If we have f and f' that satisfy

$$\int_0^{2\pi} f(\theta_1, \theta_2)d\theta_1 = \int_0^{2\pi} f'(\theta_1, \theta_2)d\theta_1 \tag{12}$$

then can we say the following?

$$f(\theta_1, \theta_2)d\theta_1 \sim f'(\theta_1, \theta_2)d\theta_1 \tag{13}$$

Put slightly differently, if ω and ω' have the same coefficients for $d\theta_2$ part (i.e., $g(\theta_1, \theta_2) = g'(\theta_1, \theta_2)$), and they have the same $a_1 = a_1'$ (i.e., satisfies (12)), does this imply $[\omega] = [\omega']$?

If $[\omega] = [\omega']$, then $a_1 = a_1'$ is satisfied is certain. You have shown it in Problem 2. What is not certain is if $a_1 = a_1'$ is satisfied, $[\omega] = [\omega']$ (assuming ω and ω' have the same $g(\theta_1, \theta_2)$).

If $a_1 = a_1'$, we have

$$\int_A \omega - \omega' = 0 \tag{14}$$

More explicitly, it means

$$\int_0^{2\pi} \left(f(\theta_1, \theta_2) - f'(\theta_1, \theta_2)\right) d\theta_1 = 0 \tag{15}$$

Assuming (14) and $g(\theta_1, \theta_2) = g'(\theta_1, \theta_2)$, if we can find ψ that satisfies

$$d\psi = \omega - \omega' \tag{16}$$

then we indeed have $[\omega] = [\omega']$

Problem 3. Show that such ψ exists by explicitly constructing one.

So, we just proved that a_1, the integration of ω over A-cycle determines the equivalence class of ω, apart from the part $g(\theta_1, \theta_2)d\theta_2$.

Similarly, if we say

$$\int_B \omega = a_2 \tag{17}$$

then, a_2 determines the equivalence class of ω, apart from the part $f(\theta_1, \theta_2)d\theta_1$.

In conclusion, a_1 and a_2 determine the equivalence class of ω. In other words, $[\omega]$ is specified by two numbers. Thus, we just proved H^1_{dR}(Torus)$= \mathbb{R}^2$.

Final comment. As promised in "Topology, the Euler characteristic, and the Gauss-Bonnet theorem," we will present yet another way to calculate the Euler characteristic. To this end, we first need to define betti number. $b_r(M)$, the rth betti number of M is defined by the exponent of $\mathbb{Z}$ in the rth homology group $H_r(M)$. For example,

$$H_0(\text{Torus}) = \mathbb{Z}^1, \qquad H_1(\text{Torus}) = \mathbb{Z}^2, \qquad H_2(\text{Torus}) = \mathbb{Z}^1 \tag{18}$$

implies

$$b_0(\text{Torus}) = 1, \qquad b_1(\text{Torus}) = 2, \qquad b_2(\text{Torus}) = 1 \tag{19}$$

Now, I will another definition of the Euler characteristic without proof. $\chi(M)$, the Euler characteristic of n-dimensional manifold M is given by

$$\chi(M) = \sum_{r=0}^{n} (-1)^r b_r(M) \tag{20}$$

Problem 4. Using this formula, check that the Euler characteristic of torus and sphere are 0 and 2, respectively.

Summary

- Homology group and cohomology group form a dual space, by the linear map

$$\langle c|\omega\rangle = \int_c \omega$$

- $b_r(M)$, the rth betti number of M, is defined by the exponent of $\mathbb{Z}$ in the rth homology group $H_r(M)$.

- $\chi(M)$, the Euler characteristic of n-dimensional manifold M, is given by

$$\chi(M) = \sum_{r=0}^{n} (-1)^r b_r(M)$$