Derivatives, velocity and acceleration

Suppose an object moves on a straight lane, and assume that there is a tree on the lane. Then, we can describe the object's position by the distance from the tree. Let this distance be s. Then, s is a function of time, since the object changes its position as time changes. To gain some insight, let's say that s is given as follows.

$$s(t) = 5t^2 \tag{1}$$

In other words, when $t = 1$ second the object was at position 5 meter. When $t = 0.4$ second the object was at position 0.8 meter and so on. See Fig. 1. It is the graph for s versus t. Now, we want to calculate the velocity of the object. However, it doesn't seem that easy, since the graph is not a straight line. If it were a straight line, its slope would be the velocity. However, here, the slope changes. Nevertheless, we can calculate the average velocity during any interval between two times. For example, the average velocity for the interval between $t = 2$ and $t = 4$ is given as follows:

$$\bar{v}_{2\sim4} = \frac{s(4) - s(2)}{4 - 2} = \frac{5 \cdot 4^2 - 5 \cdot 2^2}{4 - 2} = 30 \tag{2}$$

Here, the bar "$-$" above v means "average." The numerator is the distance the object traveled during this interval, and the denominator is the time elapsed during this interval. In other words, the average velocity is 30 m/s, because it traveled 60 m during 2 seconds. See Fig. 2. The above formula calculates the slope of the straight line that connects two points $A : (2, s(2))$ and $B : (4, s(4))$.

Now, suppose we want to calculate the object's velocity exactly at $t = 2$ instead of the average velocity. However, we do not have yet any method at our disposal at this point.

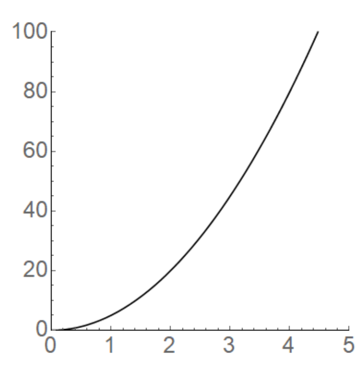

Figure 1: $s = 5t^2$

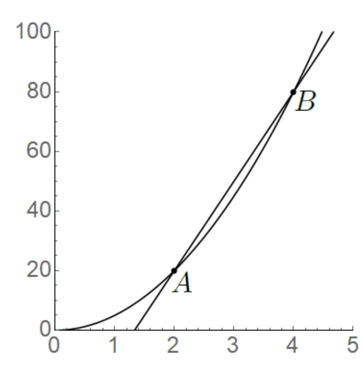

Figure 2: average velocity for $2 \le t \le 4$

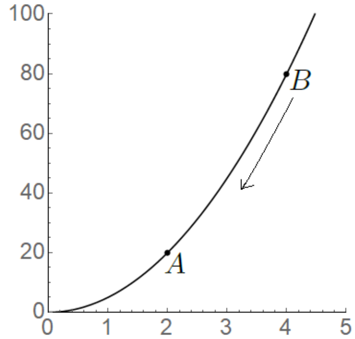

Figure 3: Pulling B closer to A

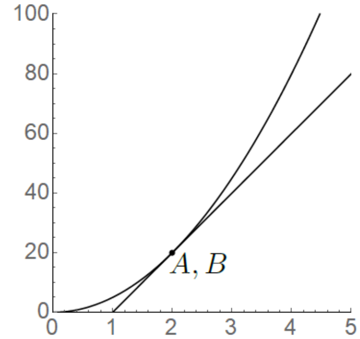

Figure 4: Velocity at $t = 2$

Nevertheless, we can still calculate the average velocity for the interval $2 \leq t \leq 2.1$ or something like that. In this case the interval is 0.1 sec. We can push this interval smaller and smaller like 0.01 sec ($2 \leq t \leq 2.01$) and 0.001 sec ($2 \leq t \leq 2.001$) and so on until the interval becomes very very small. This is the exact concept of limit. Then we will obtain something like the object's velocity exactly at $t = 2$. This corresponds to pulling B closer and closer to A (i.e. $t = 2$). See Fig. 3. When B meets A the resulting straight line that we needed to calculate the slope would be the line tangent to the point $(2, s(2))$. See Fig. 4. The straight line meets the graph $s = 5t^2$ not at two points, but only at one point designated as A, B. The slope of this straight line would be the slope of the graph $s = 5t^2$ at $t = 2$. This straight line is called "the tangent line" to the graph $s = 5t^2$ at $t = 2$. Its slope is the exact velocity when $t = 2$.

One can actually calculate the precise value for this. Let's do it. Let's calculate the object's average velocity between $t = 2$ and $t = 2 + \Delta t$. It is given as follows:

$$\begin{aligned} \bar{v}_{2\sim 2+\Delta t} &= \frac{s(t = 2 + \Delta t) - s(t = 2)}{2 + \Delta t - 2} = \frac{5(2 + \Delta t)^2 - 5 \cdot 2^2}{\Delta t} \\ &= \frac{5(4 + 4\Delta t + \Delta t^2) - 5 \cdot 4}{\Delta t} = 20 + 5\Delta t \end{aligned} \tag{3}$$

In the limit Δt goes to zero, we have the "instantaneous" velocity (i.e.t the exact velocity) 20. In other words, the instantaneous velocity when $t = 2$ is given by

$$v(t = 2) = \lim_{\Delta t \to 0} \bar{v}_{2\sim 2+\Delta t} = \lim_{\Delta t \to 0} 20 + 5\Delta t = 20 \tag{4}$$

Actually, one can calculate the instantaneous velocity at any other time using the same method. Let's calculate it when $t = t$

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{5(t + \Delta t)^2 - 5t^2}{\Delta t} = 10t \tag{5}$$

(The Greek letter Δ is pronounced as "Delta" and is used to denote a change in quantity) Now, we introduce the following notation:

$$v = \frac{ds}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} \tag{6}$$

($\equiv$ means "defined."). (We pronounce ds/dt as "ds dt.") As Δs and Δt become smaller and smaller, we call them ds and dt. In this case, we say that v is the derivative of s with respect to t. In other words, velocity is the derivative of position with respect to time. In some notations, mathematicians use $'$ (pronounced as "prime") to denote derivatives. For example, $10t = (5t^2)'$ or $v(t) = s'(t)$. Also, the process we do to find a derivative is called "differentiation."

Perhaps, an easier way of understanding ds/dt is following. s is a function of t, i.e. $s = s(t) = 5t^2$. When t becomes $t + dt$, s becomes $s + ds$. In other words,

$$s = 5t^2 \tag{7}$$

$$s + ds = 5(t + dt)^2 = 5t^2 + 10t\,dt + 5dt^2 \tag{8}$$

If you plug in (7) to the above equation, we get

$$s + ds = s + 10t\,dt + 5dt^2 \tag{9}$$

$$ds = 10t\,dt + 5dt^2 \tag{10}$$

$$\frac{ds}{dt} = 10t + 5dt \tag{11}$$

as dt is infinitesimal (i.e., infinitely small), we get

$$\frac{ds}{dt} = 10t \tag{12}$$

as expected.

Let me introduce another concept. "Acceleration" is the derivative of velocity with respect to time. If a car speeds up or slows down, we have a non-zero acceleration. However, if it goes at the same velocity, its acceleration is zero.

In our example, the acceleration is 10. Since

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{10\Delta t}{\Delta t} = 10 \tag{13}$$

(If time changes by Δt, the velocity changes by $10(t + \Delta t) - 10t = 10\Delta t$)

Differentiation is often used in physics, so it would be too bothering to obtain a derivative by using the definition such as (5). Therefore, there are easy rules to find a derivative, as much as we multiply numbers by memorizing the multiplication table, instead of adding the same number as many times as the number we need to multiply. In the next articles, we will derive these rules.

Final comment. We can differentiate only continuous functions. Let's see why this is so. Let's differentiate $s(t)$ when $t = t_0$. From (6), recall

$$\frac{ds}{dt}(t = t_0) = \lim_{\Delta t \to 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t} \tag{14}$$

Let $t_0 + \Delta t = t$. Then, the above expression becomes

$$\lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0} \tag{15}$$

Since the denominator goes to zero in this limit, the numerator must go to zero for the limit (15) to exist and have a finite value. Otherwise we have a non-zero number divided by zero. Therefore, it is necessary that

$$\lim_{t \to t_0} s(t) - s(t_0) = 0 \tag{16}$$

which is exactly the condition that the function $s(t)$ is continuous at $t = t_0$. Certainly, if a graph is "broken" at $t = t_0$, we cannot draw a unique tangent line at that point.

Problem 1. Let the position of an object at time t be given by $s = 4t + 3$. Find its instantaneous velocity at t, using a formula similar to (6). If you solve this problem correctly, you will find that its velocity doesn't depend on time. Other way of seeing this is that the slope of the graph $s = 4t + 3$ is always constant, because it's a straight line. Calculate the acceleration of this object as well using a formula similar to (13)

Problem 2. Let the position of an object at time t be given by $s = t^2 + 2t + 1$. Find its instantaneous velocity at t, using a formula similar to (6). Find the acceleration as well using a formula similar to (13).

Summary

- If you plot an object's position in y-coordinate and its time in x-coordinate, the object's velocity at a given point on the graph is given by the slope of the tangent line at that point.
- In other words, if s is an object's position and t time, the velocity v is given by

  $$v = \frac{ds}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$

  i.e., v is the derivative of s with respect to t.
- Some people express the above relation as $v(t) = s'(t)$. Here, $'$ denotes the derivative.
- The process we do to find a derivative is called "differentiation."
- Acceleration is the derivative of velocity with respect to time.