Derivatives of the trigonometric functions

In our earlier article "Addition and subtraction rules for trigonometric functions," we had:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \tag{1}$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \tag{2}$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b \tag{3}$$

$$\cos(a+b) = \cos a \cos b + \sin a \sin b \tag{4}$$

Subtracting the third equation from the first equation, we get:

$$2\cos a \sin b = \sin(a+b) - \sin(a-b) \tag{5}$$

Subtracting the fourth equation from the second equation. we get:

$$-2\sin a \sin b = \cos(a+b) - \cos(a-b) \tag{6}$$

Now plug the followings

$$a = \frac{A+B}{2}, \qquad b = \frac{A-B}{2} \tag{7}$$

into (5) and(6). Then, we get:

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \tag{8}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2} \tag{9}$$

Now, let's calculate the derivative of the sine function.

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0}\frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \frac{2\cos\frac{x+\Delta x+x}{2}\sin\frac{x+\Delta x-x}{2}}{\Delta x} \tag{10}$$

$$= \lim_{\Delta x \to 0}\cos(x+\Delta x/2)\frac{\sin(\Delta x/2)}{\Delta x/2} \tag{11}$$

Now remember that we proved the following in our earlier article.

$$\lim_{x \to 0}\frac{\sin x}{x} = 1 \tag{12}$$

Using this, (11) becomes:

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0}\cos(x+\Delta x/2)\lim_{\Delta x/2 \to 0}\frac{\sin(\Delta x/2)}{\Delta x/2} \tag{13}$$

$$= \lim_{\Delta x \to 0}\cos(x+\Delta x/2)\cdot 1 = \cos x \tag{14}$$

where in the first step we use the fact that $\Delta x/2 \to 0$ is the same condition as $\Delta x \to 0$. So, we conclude that the derivatives of $\sin x$ with respect to x is $\cos x$. Similarly, using (9), one can prove that the derivatives of $\cos x$ with respect to x is $-\sin x$.

Problem 1. Prove this.

Problem 2.

$$(x^2 \sin x)' = ? \tag{15}$$

Problem 3. (Hint[1])

$$\lim_{x\to\infty} x \sin\frac{1}{x} = ?, \qquad \lim_{x\to\infty} x \sin\frac{1}{x^2} = ? \tag{16}$$

In the rest of article, I will present an alternative way to obtain the derivatives of trigonometric functions. To this end, we will first need to prove

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0 \tag{17}$$

In our earlier article, you have proved

$$\frac{1-\cos 2\theta}{2} = \sin^2\theta \tag{18}$$

Thus, (17) becomes

$$\begin{aligned} -\lim_{x\to 0}\frac{1-\cos x}{x} &= -\lim_{2\theta\to 0}\frac{1-\cos 2\theta}{2\theta} = -\lim_{2\theta\to 0}\frac{\sin^2\theta}{\theta} \\ &= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\theta = -1\cdot 0 = 0 \end{aligned} \tag{19}$$

This completes the proof. Now, let's calculate the derivative of sine function. Using (1),

$$\begin{aligned} \frac{d}{dx}\sin x &= \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x) - \sin x}{x} = \frac{\sin x\cos\Delta x + \cos x \sin\Delta x - \sin x}{\Delta x} \\ &= \lim_{\Delta x\to 0} \sin x \frac{\cos\Delta x - 1}{\Delta x} + \cos x\frac{\sin\Delta x}{\Delta x} \\ &= \sin x \lim_{\Delta x\to 0}\frac{\cos\Delta x - 1}{\Delta x} + \cos x \lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x} \\ &= \sin x\cdot 0 + \cos x \cdot 1 = \cos x \end{aligned} \tag{20}$$

This completes the proof.

Problem 4. Using (2), (12) and (17), prove

$$\frac{d}{dx}\cos x = -\sin x \tag{21}$$

Summary

- $\dfrac{d(\sin x)}{dx} = \cos x$
- $\dfrac{d(\cos x)}{dx} = -\sin x$

[1] Reexpress the formulas in terms of variable $y = 1/x$.