Determinant of 2×2 matrices

In the earlier article "Matrix inverses," I noted that the solution of the following equation:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{1}$$

is given by the following:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{2}$$

where $\det A$ is given by the following:

$$\det A = a_{11}a_{22} - a_{21}a_{12} \tag{3}$$

Given this, I want to ask "When does the solution not exist?" We all know that one cannot divide a certain quantity by zero. Therefore, it is easy to see from (2) that this condition amounts $\det A = 0$. The point is that the ratio between a_{11} and a_{12} is the same as a_{21} and a_{22}. Therefore, if you try to eliminate one of the variables x_1 and x_2 by multiplying one of the equations by appropriate factor and subtract this from the other equation you get an equation like "$0x + 0y =$something." Let me give you an explicit example so that you can understand better what I am saying:

Let's say

$$\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix} \tag{4}$$

In other words:

$$6x_1 + 4x_2 = 10 \tag{5}$$

$$3x_1 + 2x_2 = 4 \tag{6}$$

Multiplying (6) by 2 and subtracting from (5) yields the following:

$$0x_1 + 0x_2 = 2 \tag{7}$$

which obviously doesn't have any solutions. As the left-hand side is always 0 which can never equal to 2.

However, in fact, there are cases that the solutions exist even though the determinant is zero. The condition that the determinant is zero implies that one can obtain "$0x_1 + 0x_2$" from the standard rule of solving simultaneously equations, but it never tells about the right-hand side of "$0x_1 + 0x_2$" which in our example is 2. Therefore, if the right-hand side is zero, we can have solutions, and in fact infinitely many.

Now, to learn the property of determinant better, let's consider the case that $y_1 = y_2 = 0$. When the determinant is not zero, the solution will be always $x_1 = x_2 = 0$. We can easily see this from (2). However, when determinant is zero, we can have solutions other than $x_1 = x_2 = 0$. As the ratio between a_{11} and a_{12} is equal to a_{21} and a_{22}, any x_1 and x_2 that satisfy (8) will satisfy (9).

$$a_{11}x_1 + a_{12}x_2 = 0 \tag{8}$$
$$a_{21}x_1 + a_{22}x_2 = 0 \tag{9}$$

For example, if $6x_1 + 4x_2 = 0$, $3x_1 + 2x_2 = 0$ is automatically satisfied. Also, notice that there are infinitely many such solutions as (8) has infinitely many solutions.

Therefore, in this article we learned that for 2×2 matrix "A" and 2×1 matrix "x" $Ax = 0$ has non-zero solutions if and only if $\det A = 0$. This can be generalized to $n \times n$ matrix where n is a bigger number, if we can define the corresponding determinant of $n \times n$ matrix appropriately. For example, (10) has a non-zero solution, if and only if (11) is satisfied.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 0 \tag{11}$$

Some books use the following notation for determinant:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Notice that the brackets of matrix are replaced by vertical bars to denote the determinant.

If you have read our earlier article "System of linear equations, part II: three or more unknowns," you may notice that the condition that (10) has a non-zero solution is equivalent to the condition that the equations (10) are linearly dependent. In any case, we will show you how one can define determinant for $n \times n$ matrix for any n in the next article.

Summary

- $Ax = 0$, where A is a matrix, has a non-zero solution x if and only if $\det A = 0$.