The determinant and its geometric interpretation

In the earlier article, I introduced the determinant and its algebraic interpretation. In this article, I will explain the geometric interpretation of the determinant. To this end, I will first consider the relevant case of 2×2 matrix. Then, I will simply note that one can generalize it to the higher-dimensional matrices.

Consider the two vectors $\vec{v}_1$ and $\vec{v}_2$ as in the first figure. I will show that the area of the parallelogram spanned by these two vectors is given by the following formula:

$$\det(\vec{v}_1, \vec{v}_2) = \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix} \tag{1}$$

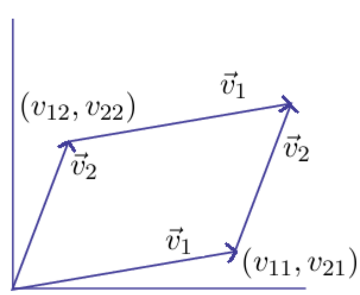
where

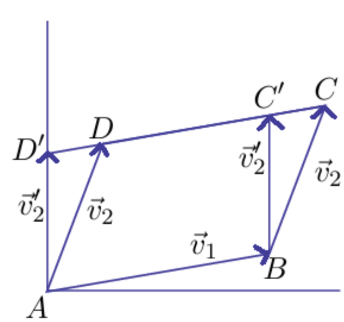
$$\vec{v}_1 = v_{11} i + v_{21} j \tag{2}$$

$$\vec{v}_2 = v_{12} i + v_{22} j \tag{3}$$

as in the figure.

To show that the determinant is indeed the area of the parallelogram, I will deform the parallelogram into another parallelogram and then this new parallelogram, in turn, to a rectangle in such a way that the area doesn't change. Along this process I will show that the determinant of the corresponding matrix for the new parallelogram and the determinant of the corresponding matrix for the rectangle remain to be the same as the determinant of the matrix for the original parallelogram. Finally, I will show that the area of the rectangle is simply given by the determinant of the corresponding matrix for the rectangle, thereby completing the proof.





Now, here comes the trick. See the second figure. It is easy to see that the area of the parallelogram spanned by $\vec{v}_1$ and $\vec{v}_2'$ is the same as the area of the original parallelogram, namely the one spanned by $\vec{v}_1$ and $\vec{v}_2$. In other words, the area of the parallelogram $ABC'D'$ is the same as that of the parallelogram $ABCD$. What is the vector $\vec{v}_2'$ in terms of $\vec{v}_1$ and $\vec{v}_2$?

It is easy to see that it has to be given by the following formula:

$$\vec{v}_2' = \vec{v}_2 - c\vec{v}_1 \tag{4}$$

for some suitable c.

It is easy to see that this is the case since the line segment $D'D$ is parallel to the line segment AB, so the vector DD should be a scalar multiple of the vector AB. Notice that c is simply given by the ratio between the length of $D'D$ and the length of AB.

Notice also that the determinant remains to be the same. The determinant for the corresponding matrix for the new parallelogram v is given by the following:

$$\begin{aligned}
\det v' &= \det(\vec{v}_1, \vec{v}_2') = \det(\vec{v}_1, \vec{v}_2 - c\vec{v}_1) \\
&= \det(\vec{v}_1, \vec{v}_2) - c\det(\vec{v}_1, \vec{v}_1) = \det(\vec{v}_1, \vec{v}_2) \\
&= \det v
\end{aligned} \tag{5}$$

Now comes the second part of the operation. See the third figure. It is easy to see that the area of the rectangle spanned by $\vec{v}_1'$ and $\vec{v}_2'$ is the same as the area of our second parallelogram, namely the one spanned by $\vec{v}_1$ and $\vec{v}_2'$. In other words, the area of the rectangle $AB''C''D'$ is the same as that of the parallelogram $ABC'D'$. What is the vector $\vec{v}_1'$ in terms of $\vec{v}_1$ and $\vec{v}_2'$?

It is easy to see that it has to be given by the following formula:

$$\vec{v}_1' = \vec{v}_1 - c'\vec{v}_2' \tag{6}$$

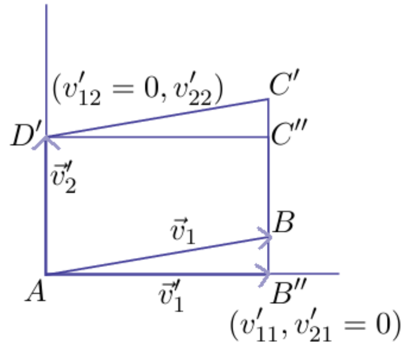
for some suitable c'.

It is easy to see that this is the case since the line segment $B''B$ is parallel to the line segment AD', so the vector $B''B$ should be a scalar multiple of the vector AD'.

Again, the determinant remains to be the same. Let's check this. The determinant for the corresponding matrix for the rectangle v'' is given by the following:

$$\begin{aligned}
\det v'' &= \det(\vec{v}_1', \vec{v}_2') = \det(\vec{v}_1 - c'\vec{v}_2', \vec{v}_2') \\
&= \det(\vec{v}_1, \vec{v}_2') - c'\det(\vec{v}_2', \vec{v}_2') = \det(\vec{v}_1, \vec{v}_2') \\
&= \det v'
\end{aligned} \tag{7}$$

Combining with our earlier result, we see that $\det v = \det v''$. As the final step let's see that $\det v''$ is indeed the area of the rectangle. It is obvious from the picture that v_{12}' and



v_{21} have to be zero. Therefore, we obtain:

$$\det v'' = \det(\vec{v}'_1, \vec{v}'_2) = \begin{vmatrix} v'_{11} & 0 \\ 0 & v'_{22} \end{vmatrix} = v'_{11} \begin{vmatrix} 1 & 0 \\ 0 & v'_{22} \end{vmatrix} = v'_{11} \cdot v'_{22} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = v'_{11} \cdot v'_{12} \tag{8}$$

where we have used the multilinearity of the determinant and the fact that the determinant of identity matrix is 1.

Now notice that this is also the area of the rectangle. This completes the proof. (Strictly speaking, $\det v''$ could be negative, since v'_{11} or v_{12} could be negative. In such cases the absolute value of the determinant gives the area of the rectangle.)

For the determinant of the higher-dimensional matrices, it has the interpretation as the higher dimensional analogue of volume of the parallelepiped spanned by the corresponding vectors. The proof is the simple generalization of what we have done in this article.

Final comment. As I just said, for any $n \times n$ matrix, we can express its determinant by the process we have done in this article for 2×2 matrix. Then, the final result is expressed as a product of some numbers multiplied by the determinant of identity matrix. Thus, this procedure uniquely determines the determinant, as advertised in our last article. There can be no two values for the determinant evaluated this way.

Summary

- Determinant calculates the volume of parallelepiped spanned by vectors.