Differential and infinitesimal change

In earlier articles, we used the expressions such as dx, dy and so on. Here, d is called "differential." In this article, we will review this concept once again, and get more used to it. Remember how dy/dx was defined. It was defined as $\Delta y/\Delta x$ in the limit Δx goes to 0. As Δ denotes the change, we can regard d the differential as infinitesimal change, and dy/dx as the ratio of infinitesimal change.

Given this, let's try to find some rules the differential satisfies. First, from the followings

$$\Delta(f+g) = (f+\Delta f+g+\Delta g)-(f+g) = \Delta f+\Delta g \tag{1}$$
$$\Delta(f-g) = (f+\Delta f-g-\Delta g)-(f-g) = \Delta f-\Delta g \tag{2}$$

we have

$$d(f+g) = df+dg, \qquad d(f-g) = df-dg \tag{3}$$

Furthermore, if c is a constant, from $\Delta(cf) = c\Delta f$, we have

$$d(cf) = c\,df \tag{4}$$

Now, let's derive Leibniz rule.

$$\Delta(fg) = (f+\Delta f)(g+\Delta g)-fg = (\Delta f)g+f\Delta g+\Delta f\Delta g \tag{5}$$

Given that the last term is much smaller than the first two terms in the limit $\Delta f, \Delta g \to 0$, we can neglect the last term. Then, we have

$$d(fg) = (df)g+f(dg) \tag{6}$$

Finally, the following identity

$$\Delta f = \frac{\Delta f}{\Delta x}\Delta x \tag{7}$$

implies

$$df = \frac{df}{dx}dx \tag{8}$$

Now, let's apply the concept of differential. From Leibniz rule, we have

$$d(x^2) = d(xx) = (dx)x+x(dx) = 2x\,dx \tag{9}$$

$$d(x^3) = d(xx^2) = (dx)x^2+xd(x^2) = x^2\,dx+x\,2xdx = 3x^2\,dx \tag{10}$$

We can apply Leibniz rule repeatedly this way. Then, by induction, we can prove

$$d(x^n) = nx^{n-1}\,dx \tag{11}$$

for a positive integer n. Of course, we could alternatively arrive at the above formula as follows.

$$d(x^n) = \frac{d(x^n)}{dx}dx = nx^{n-1}\,dx \tag{12}$$

Now, as promised in our earlier article "Derivatives of the polynomials," let's prove (11) for any n. To this end, note $d(\ln x)/dx = 1/x$ implies

$$d(\ln x) = \frac{dx}{x} \tag{13}$$

Given this, observe $\ln(x^n) = n \ln x$. Taking differential on the both-hand sides, we get

$$d(\ln(x^n)) = n\,d(\ln x) \tag{14}$$

$$\frac{d(x^n)}{x^n} = n\frac{dx}{x} \tag{15}$$

which is precisely (11).

Now, let's apply the concept of differential in our everyday lives. Suppose we have a rectangular with side a, b, c. Then, its volume is given by $V = abc$. If we increase a by 0.1%, b by 0.3%, and decrease c by -0.05%, what is the rough change in the volume in percentage? As the changes are very small and we want to know the rough change, we will use the concept of differential. We have

$$V = abc \tag{16}$$

$$\ln V = \ln a + \ln b + \ln c \tag{17}$$

$$\frac{dV}{V} = \frac{da}{a} + \frac{db}{b} + \frac{dc}{c}, \qquad \text{i.e.,} \quad \frac{\Delta V}{V} = \frac{\Delta a}{a} + \frac{\Delta b}{b} + \frac{\Delta c}{c} \tag{18}$$

So, we see that the relative change of the volume is the sum of the relative change of the sides. Therefore, we have 0.1%+0.3%−0.05%=0.35%. 0.35% is the answer. Actually, the exact answer is 0.35009985%; our approximation is very good.

Problem 1. Let $S = 5a^2b^3c^4$. Then, express dS/S in terms of da/a, db/b and dc/c.

Problem 2. Let $x = \sqrt{y}/z$. Then, express dx/x in terms of dy/y and dz/z.

Summary

- $df = \frac{df}{dx}dx$
- $d(x^n) = nx^{n-1}dx$
- $d(\ln x) = \frac{dx}{x}$
- Let $V = abc$, then as $\ln V = \ln a + \ln b + \ln c$,

$$\frac{dV}{V} = \frac{da}{a} + \frac{db}{b} + \frac{dc}{c}$$