Differential equations

Suppose that there are initially N_0 number of "A" atoms, and every second λ proportion of them decay to become other atoms. Then, the question is, how many of A atoms are left after t seconds? To solve this problem, let's say that the number of A atoms after t seconds is given by $N(t)$. Then, after another second, the number of A atoms will decrease by λN. In other words,

$$\frac{dN}{dt} = -\lambda N \tag{1}$$

Now, all we need to do is solve this equation. Noticing that one gets a form of exponential function again, if one differentiates it, we try $N = be^{at}$ as a solution to the above equation. Plugging this in, the above equation becomes:

$$abe^{at} = -\lambda be^{at} \tag{2}$$

Therefore, we conclude $a = -\lambda$. So, we have $N = be^{-\lambda t}$. Furthermore, as we have $N(t = 0) = N_0$, the final answer is given by:

$$N = N_0 e^{-\lambda t} \tag{3}$$

We say that (3) is a solution to the differential equation (1). Notice also that we cannot determine b in $N = be^{-\lambda t}$, if we didn't have the additional information, namely the initial value of the number of atoms. Conditions like this which is necessary to determine unknown constants in the solution to differential equations are called "initial conditions."

Now, let us pose a question. Is there a more systematic way to obtain a solution to a differential equation? We will introduce one. (1) implies:

$$\begin{aligned}
\frac{dN}{N} &= -\lambda dt && (4)\\
\ln N + C_1 &= -\lambda t + C_2 \\
N &= e^{-\lambda t + C_2 - C_1} \\
N &= be^{-\lambda t} && (5)
\end{aligned}$$

where from the first line to the second line, we integrated, and from the third line to the fourth line we used the fact that C_1 and C_2 are arbitrary constants so we defined another arbitrary constant simply by

$$b \equiv e^{C_2 - C_1} \tag{6}$$

Yet, another example:

$$
\begin{aligned}
\frac{dy}{dx} &= xy & (7)\\
\frac{dy}{y} &= xdx & (8)\\
\ln y &= x^2/2 + C & (9)\\
y &= Ae^{x^2/2} & (10)
\end{aligned}
$$

However, consider this example

$$
\begin{aligned}
\frac{dy}{dx} &= yx + x^2 & (11)\\
dy &= (yx + x^2)dx & (12)
\end{aligned}
$$

So, we are stuck here; our trick to separate all y-dependence to the left hand-side and all x-dependence to the right hand-side doesn't work. This trick is called "separation of variables." If such a trick doesn't work as in this example, one would need to use other methods, in some cases, even numerical methods using computer; I personally don't know the analytic (i.e. closed form expressed by mathematical formulas rather than numerical form) solution to above equation.

Finally, let me introduce another common example that cannot be solved by the separation of variable.

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{13}$$

Noticing that sine functions and cosine functions pick up negative signs if differentiated twice, let's try $x = A\cos st + B\sin st$. Then, we have:

$$\frac{d^2x}{dt^2} = -s^2(A\cos st + B\sin st) = -s^2 x \tag{14}$$

Therefore, $s = \omega$. We conclude:

$$x = A\cos\omega t + B\sin\omega t \tag{15}$$

where A and B has to be determined from initial conditions. When the separation of variable method doesn't work, it is often the case that one has to rely upon guess such as this one.

Problem 1. In (3), how long does it take for the number of atoms to be halved? This is known as "half-life." Convince yourself that the answer to this question is independent of the initial condition. In other words, the time it takes for the number of atoms to be halved doesn't depend on how many atoms there were initially. Decay such as this one is called "exponential decay."

Problem 2. Let's say that the half-life of a certain atom is 100 sec. Then, how long does it take for the number of such atoms to be quartered? How about eighthed? (Hint[1])

Now, we will introduce "mean lifetime" or simply "lifetime." It's the mean life-time for which atoms can survive before being decayed. To obtain the mean life-time, let's first obtain the probability density function for the atoms to decay. It should be proportional to $e^{-\lambda t}dt$. Thus, we can write

$$P(t) = Ce^{-\lambda t} \tag{16}$$

Then, C can be obtained by the normalization condition

$$1 = \int_0^\infty P(t)dt \tag{17}$$

which implies

$$C = \frac{1}{\int_0^\infty e^{-\lambda t}dt} \tag{18}$$

Thus, the mean lifetime is given by (**Problem 3.** Check!)

$$\langle t \rangle = \int_0^\infty tP(t)dt = \frac{\int_0^\infty te^{-\lambda t}dt}{\int_0^\infty e^{-\lambda t}dt} = \frac{1}{\lambda} \tag{19}$$

Problem 4. We have seen that a falling object with mass m receives a force equal to mg. However, this is only so when the air friction is ignored. In reality, the air friction which greatly reduces the falling speed of an object can't be ignored. If there were no air friction, the rain would drop at an enormous speed which would kill anyone without an umbrella.

In this problem, we will assume that the air friction is proportional to the speed of the falling object, and the proportionality constant is given by b. This is reasonable considering that the bigger the speed of the object the bigger the air friction.[2] Then, we can write the following differential equation:

$$m\frac{dv}{dt} = mg - bv \tag{20}$$

Here, we chose the sign of v to be positive when the object falls downward. Therefore, the sign in front of mg is positive, and not negative as before.

[1] Being quartered is the same thing as being halved twice. Being eighted is the same thing as being halved thrice.

[2] In reality, this is more subtle. According to Wikipedia, air friction is proportional to the velocity for a laminar flow and the velocity squared for a turbulent flow. I don't know what a laminar flow and a turbulent flow are, but in any case our assumption catches all the qualitative features of air friction in general.

On the other hand, the sign in front of bv is negative as the air friction is in the opposite direction of moving velocity.

Before actually solving this equation, let's discuss some qualitative features of the solution without solving. If initially, the falling speed was zero, the initial acceleration would be g as $v = 0$. However, as it gains speed the acceleration would decrease as the total force $mg - bv$ is smaller for bigger v. After a lot of time is elapsed, the total force $mg - bv$ will be further reduced and become (almost) zero. The speed of object at such a moment can be obtained by solving $mg - bv = 0$. We obtain $v = mg/b$. As the total force is now zero, it will fall at the constant speed $v = mg/b$. This is known as "terminal velocity."

Find an explicit solution for (20) and verify all the qualitative features of the solution explained in the last paragraph.(Hint[3]) Also, describe qualitatively what will happen if the initial falling speed was bigger than the terminal velocity.

Problem 5. Show that the following is also the solution to (13).

$$x = C\cos(\omega t - \theta_0) \tag{21}$$

By using subtraction rule for trigonometric function, show the following:

$$C = \sqrt{A^2 + B^2}, \qquad \tan\theta_0 = \frac{B}{A} \tag{22}$$

Problem 6. This problem deals with another trick to solve (13). Let's try a solution of a form $x = De^{\alpha t}$. Obtain α. You will get two αs. Then, a general solution can be written as something like:

$$x = D_1 e^{\alpha_1 t} + D_2 e^{\alpha_2 t} \tag{23}$$

as the sum of two solutions is also a solution. Show that this solution is equivalent to (15). (Hint[4])

Problem 7. Solve the following. (Hint[5])

$$\frac{d^2 y}{dt^2} = -\omega^2 y + k \tag{24}$$

Summary

- Differential equations are equations that involve differentiation.
- Some differential equations can be solved by the trick called "separation of variable."

[3]You can either use the separation of variables method or introduce a new variable $u \equiv v - mg/b$. Expressing the differential equation using this variable, the equation is precisely in the form of (1) which you know how to solve.

[4]Use Euler's formula. Express A and B explicitly in terms of D_1 and D_2.

[5]If we let $x \equiv y - k/\omega^2$, the equation is precisely in the form of (13).

- The solution to

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

is of the form

$$x = A\cos\omega t + B\sin\omega t$$