The power of dimensional analysis

This article is inspired by a problem in the entrance exam for the Korea Advanced Institute of Science and Technology in 1993. Consider the pendulum in Figure 1. Suppose you gently release the ball with mass m when the string attached to the ball makes an angle of θ with the vertical line. If the length of string holding the ball is l, the mass of string is negligible, and the gravitational constant is g, which of the following is true for the tension of the string when the ball reaches the lowest point?

1) If l is increased, the tension increases.
2) If l is increased, the tension decreases.
3) Even if l is increased, the tension remains the same.

The answer is 3). Now, let's think about why this must be the answer. Clearly, the tension T must be a function of l, θ, m, and g. Now notice that T has to have the dimension of force (which can be expressed in units as $\mathrm{kg} \cdot \mathrm{m/s^2}$). So, we have to make a quantity that has the dimension of force out of l, θ, m, g. The unit of l is meters, θ is dimensionless, the unit of m is kilograms, and the unit of g is $\mathrm{m/s^2}$. So, to make something that has the dimension of force, we have to first multiply m by g. Then, we notice that we can further multiply a dimensionless quantity, say $f(\theta)$, to denote a function of θ which is dimensionless. (Functions of θ, such as cosine and sine, are dimensionless, since they are ratios of the sides of a triangle. The angle θ itself is also dimensionless since the definition of an angle in radians is simply the ratio between the arc and the radius.) On the other hand, we notice that there is no place for l to budge in without changing the dimensions to something other than force. So the answer is independent of l, and the answer is 3). I want to point out that we arrived at this conclusion without actually calculating the tension. This method of approaching a problem solely by checking dimensions is called "dimensional analysis" and is a very powerful tool in wide-ranging areas of physics.

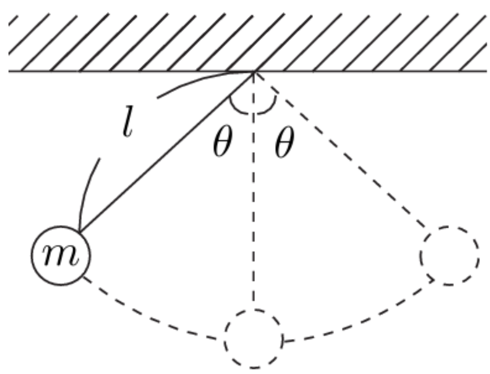

Figure 1: a pendulum