Dimensionful quantity and dimensionless quantity

In everyday life, we use numbers all the time. Molly weighs 59 kg and she is 170 cm tall. Country A's GDP is one-half that of country B's GDP. The airplane flies at 900 km/h. Jonathan runs 1.5 times faster than Youngsub. Jane has three brothers. All these numbers can be divided into two categories: dimensionful quantities and dimensionless quantities. 59 kg, 170 cm, 900 km/h are examples of dimensionful quantities and half, 1.5, three are examples of dimensionless quantity.

Dimensionful quantity and dimensionless quantity are fundamentally different. The numerical value of a dimensionful quantity depends on the unit we choose, whereas the value of a dimensionless quantity never changes when the unit is changed. For example, Molly weighs 130 pounds, and she is 5 feet 7 inches tall. Likewise, the airplane flies at 560 miles per hour. However, country A's GDP is half of country B's GDP, no matter whether you calculate it by dollars, cents, or euros. The same can be said about Jonathan's running speed. Let's say he runs 9 m/s while Youngsub runs 6 m/s so that he runs 1.5(=9÷6) times faster than Youngsub. Then, his speed is 32.4 km/h while Youngsub's speed is 21.6 km/h. Jonathan still runs 1.5(=32.4÷21.6) times faster than Youngsub, even if we change the unit. Nor can we change the number of Jane's brothers by changing the unit, because we cannot change the unit. Brothers are brothers. Can we say she had six half-brothers instead of saying she has three brothers? Then, it would mean something different.

Another important point is that addition and subtraction are only meaningful when the two quantities have the same dimension. Molly weighs 59 kg and her luggage weighs 20 pounds. How much do they weigh in total? Since both kg and pounds are units for weight (or more precisely, mass), we can add them after converting the units. As 1 pound is about 0.45 kg, 20 pounds is about 9 kg. Thus, Molly and her luggage weigh 68 kg in total. We can obtain the same results in pounds. As 59 kg is 130 pounds, Molly and her luggage weigh 150(=130+20) pounds. Indeed, this makes sense because 68 kg is 150 pounds. However, we cannot say she and her luggage weigh 60(=59+1) kg, because she weighs 59 kg and she has one piece of luggage. Nor can we say that her height and weight are altogether 230(=170 cm+60 kg) cm. It would have a different value, if we used a different unit like this: Her height and weight are altogether 300(=170 cm+130 pounds) cm. One calculation gives 230 cm, while another gives 300 cm. Since the result depends on the unit chosen, the quantity is meaningless.

However, not everyone realizes this. When I was in elementary school, there was an incorrect problem on a math exam. I do not remember the exact number or the exact formula, but it was something like this.

$$50 \text{ cm} + 1 \text{ m} + 40 \text{ cm} \div 2 \text{ m} \tag{1}$$

You can add the first two numbers, because they are both lengths. It yields 150 cm(=50 cm+100 cm) or equivalently 1.5 m(=0.5 m+ 1 m). On the other hand, you cannot add the last term to the first two numbers, because it is dimensionless; 40 cm÷2 m is 0.2(=40 cm÷200 cm). You cannot add the dimensionless number 0.2 to 150 cm (or 1.5 m). It would give 150.2 cm or 1.7 m, which do not make sense at all. Even though nobody had taught me before that you cannot add numbers if they have different dimensions, I knew that it was a wrong problem, and pointed it out to my math teacher. However, he didn't understand what I meant and said something like if centimeters are converted to meters, they can be calculated, even if their units are different.

I later found out that this kind of mistake was much more common than I thought. In my university physics class, many students wrote meaningless expressions. I asked them to calculate the speed of something in an exam, and as its answer, a student wrote

$$v = \sqrt{\frac{c^2}{c^2+1}}. \tag{2}$$

Here, v denotes the speed being sought and c denotes the speed of light. This answer doesn't make sense, because c^2 has the dimensions of speed squared, whereas 1 is dimensionless. The correct answer is

$$v = \sqrt{\frac{c^2}{1+1}} = \frac{c}{\sqrt{2}}. \tag{3}$$

As a side note, dimensionless constants are particularly important in physics. Dimensionful quantity depends on the unit we use while dimensionless constant does not depend on the unit, as we mentioned. Dimensionful quantities are human-made, whereas dimensionless constants are God-given.

Summary

- Numbers can be divided into two categories: dimensionful quantities and dimensionless quantities.
- Two quantities can be added or subtracted only if they have the same dimension.