Dirac delta function

What would be the wave function $\phi(x)$ that corresponds to $|x_0\rangle$, the eigenvector of position operator $\hat{x}$ with the eigenvalue x_0? Certainly, if you measure the position of the object that corresponds to this wave function, you will get x_0 with 100% certainty, and you will get other values with 0% certainty. Therefore, it is clear that $\phi(x)$ is zero unless $x = x_0$. With this intuition in our mind, let's calculate $\phi(x)$ mathematically.

$$|x_0\rangle = \int_{-\infty}^{\infty} dx|x\rangle\langle x|x_0\rangle = \int_{-\infty}^{\infty} \langle x|x_0\rangle|x\rangle dx \tag{1}$$

In other words, $\phi(x) = \langle x|x_0\rangle$. However, as we know that the position operator is a Hermitian matrix, which in turn implies that the eigenvectors are orthogonal to each other unless they have the same eigenvalues, it is easy to see that $\langle x|x_0\rangle = 0$ unless $x = x_0$ (i.e. $x - x_0 = 0$). Thus, we recover our earlier argument. Notice also that $\langle x|x_0\rangle$ is a function of $(x - x_0)$ only. Therefore, for a certain function $\delta(x - x_0)$, we can write as follows:

$$\langle x|x_0\rangle = \delta(x - x_0) \tag{2}$$

This function is called Dirac delta function. Now, let's determine its value when $x - x_0 = 0$. To this end, let's return to (1). We have:

$$|x_0\rangle = \int_{-\infty}^{\infty} \delta(x - x_0)|x\rangle dx \tag{3}$$

Since $\delta(x - x_0) = 0$ if $x \neq x_0$, the only contribution to the integral comes when $x = x_0$. Therefore, the right-hand side of the above equation becomes $c|x_0\rangle$ for some c. However, from the left-hand side, it is apparent that c should be 1. Also, notice the following for infinitesimal ϵ

$$\begin{aligned} |x_0\rangle &= \int_{x=x_0-\epsilon}^{x=x_0+\epsilon} \delta(x - x_0)|x\rangle dx \\ &= \int_{x=x_0-\epsilon}^{x=x_0+\epsilon} \delta(x - x_0)|x_0\rangle dx \end{aligned} \tag{4}$$

Therefore, we conlcude:

$$\begin{aligned}
1 &= \int_{x=x_0-\epsilon}^{x=x_0+\epsilon} \delta(x-x_0)dx \\
&= \int_{x-x_0=-\epsilon}^{x-x_0=\epsilon} \delta(x-x_0)dx \\
&= \int_{-\epsilon}^{\epsilon} \delta(x)dx \qquad (5) \\
1 &= \int_{-\infty}^{\infty} \delta(x)dx \qquad (6)
\end{aligned}$$

where we used the change of variable in going from the second line to the third line, and we used that $\delta(x) = 0$ unless $x = 0$ in going from the third line to the fourth line. In conclusion, Dirac delta function is defined by two conditions. First, $\delta(x) = 0$ unless $x = 0$. Second, the formula (6).

Let me also remark that the eigenvector of position operator $|x\rangle$ is not normalized. Otherwise $\langle x|x\rangle$ would have been 1, which is not the case. If $\langle x|x\rangle = 1$ (i.e.$\delta(x-x=0) = 1$ were true, in light of (5), this would imply:

$$\begin{aligned}
1 &= \int_{-\epsilon}^{\epsilon} 1dx = 0 \\
&= 2\epsilon \\
&= 0 \qquad (7)
\end{aligned}$$

Actually, it is easy to see that (5) implies that $\delta(0) = \infty$, since integrating this on an infinitesimal interval yielded a finite value. In conclusion, the eigenvector of the position operator is not normalized, as eigenvalues are continuous. In fact, enforcing its normalization is not natural.

Finally, let's show you another property of Dirac delta function.

$$f(x_0) = \int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx \qquad (8)$$

The reasoning is as follows. As $\delta(x-x_0)$ is non-zero only when $x = x_0$, the integration picks up the contribution only when $x = x_0$. So, as before, we get:

$$\begin{aligned}
&\int_{x_0-\epsilon}^{x_0+\epsilon} f(x)\delta(x-x_0)dx \qquad (9) \\
&= \int_{x_0-\epsilon}^{x_0+\epsilon} f(x_0)\delta(x-x_0)dx \qquad (10) \\
&= f(x_0)\int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x-x_0)dx \qquad (11) \\
&= f(x_0) \qquad (12)
\end{aligned}$$

where in the last step we used (5)

We will revisit Dirac delta function when we talk about the relation between position basis and momentum basis in a later article.

Problem 1. Evaluate the followings:

$$\int_{-\infty}^{\infty} dx(x^2+4x)\delta(x-4), \qquad \int_{-3}^{3} dx(x^2-4x)\delta(x+4) \tag{13}$$

Problem 2. Recall how, in our earlier article on Dirac's bra-ket notaion, we derived $\langle u|v\rangle = \sum_i u_i v_i$ if we have $|v\rangle = \sum_i v_i|e_i\rangle$ and $\langle u| = \sum_i u_i\langle e_i|$, where $\langle e_i|e_j\rangle = \delta_{ij}$. Using $\langle x|x'\rangle = \delta(x-x')$, derive similarly

$$\langle a|b\rangle = \int a^*(x)b(x)dx \tag{14}$$

where

$$|a\rangle = \int a(x)|x\rangle dx, \qquad |b\rangle = \int b(x)|x\rangle dx \tag{15}$$

You should not use the completeness relation or $a(x) = \langle x|a\rangle$.

Problem 3. Convince yourself of the followings:

$$\delta(-x) = \delta(x), \qquad \delta(3x) = \frac{1}{3}\delta(x), \qquad \delta(-3x) = \frac{1}{3}\delta(x) \tag{16}$$

Problem 4. Some books use the following notation: $\delta(x,y) \equiv \delta(x-y)$. Given this, calculate the following:

$$\int_{-\infty}^{\infty} f(y)\delta(x,y)dy \tag{17}$$

Problem 5. Calculate the following:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz \tag{18}$$

Some books use the following notation: $\delta^3(\vec{x}) \equiv \delta(x)\delta(y)\delta(z)$.

Problem 6. Calculate the following:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-f(y,z))\delta(y-g(z))\delta(z)dxdydz \tag{19}$$

Problem 7. Remember that we have seen in the last article that $\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle$. To complete the calculation, show the following:

$$\langle\psi|\hat{x}|\psi\rangle = \int_{-\infty}^{\infty} dx\psi^*(x)x\psi(x) \tag{20}$$

where $\psi(x) = \langle x|\psi\rangle$. (Hint[1])

[1]Remember what we have done in the last article. Use also (**??**) and (**??**).

Problem 8. Heaviside step function θ is defined as follows:

$$\theta(x) = \int_{-\infty}^{x} \delta(x)dx \tag{21}$$

Given this, evaluate the followings:

$$\theta(-50), \qquad \theta(-30), \qquad \theta(5), \qquad \theta(0.3) \tag{22}$$

Problem 9. Evaluate the following (Hint[2]):

$$\int_{-\infty}^{\infty} \delta(x^2 - 4x)(x+2)dx \tag{23}$$

Problem 10. (Challenging!) Show that

$$\frac{d\delta(x)}{dx} = -\frac{\delta(x)}{x} \tag{24}$$

Summary

- If $|x\rangle$ is an eigenvector of position operator $\hat{x}$ with eigenvalue x, it satisfies the normalization condition $\langle x|x_0\rangle = \delta(x - x_0)$.
- Dirac delta function $\delta(x)$ is 0 for $x \neq 0$, and infinite when $x = 0$. It satisfies $1 = \int_{-\infty}^{\infty} \delta(x)dx$.
- $$f(x_0) = \int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx$$

[2] Use $x^2 - 4x = x(x-4)$, and think about when the Dirac delta function is non-zero. Then, use similar formulas to the ones in Problem 3.