Divergence and Stoke's theorem

1 Stoke's theorem

Suppose you have a very small three-dimensional region as follows

$$x_0 \le x \le x_0 + \Delta x, \quad y_0 \le y \le y_0 + \Delta y, \quad z_0 \le z \le z + \Delta z \tag{1}$$

and you want to calculate the flux coming out of this region given the following vector field

$$\vec{U} = U_x(x, y, z)\hat{x} + U_y(x, y, z)\hat{y} + U_z(x, y, z)\hat{z} \tag{2}$$

We can do this by computing the contributions from U_x, U_y and U_z separately and add them up. So, let's calculate the contribution from U_x first. See Fig. 1. You see that the contribution should be

$$\int_{z_0}^{z_0+\Delta z} \int_{y_0}^{y_0+\Delta y} U_x(x_0 + \Delta x, y, z)dydz - \int_{z_0}^{z_0+\Delta z} \int_{y_0}^{y_0+\Delta y} U_x(x_0, y, z)dydz$$

$$= \int_{z_0}^{z_0+\Delta z} \int_{y_0}^{y_0+\Delta y} (U_x(x_0 + \Delta x, y, z) - U_x(x_0, y, z))dydz \tag{3}$$

Now, using the Taylor expansion, we have:

$$U_x(x_0 + \Delta x, y, z) \approx U_x(x_0, y, z) + \frac{\partial U_x}{\partial x}(x_0, y, z)\Delta x \tag{4}$$

Then, (3) becomes

$$\lim_{\Delta y, \Delta z \to 0} \int_{z_0}^{z_0+\Delta z} \int_{y_0}^{y_0+\Delta y} \frac{\partial U_x}{\partial x}(x_0, y, z)\Delta x dydz$$

$$= \lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\partial U_x}{\partial x}(x_0, y_0, z_0)\Delta x \Delta y \Delta z \tag{5}$$

We can do the similar calculation for U_y as well. See Fig. 1. Then, we get:

$$\frac{\partial U_y}{\partial y}(x_0, y_0, z_0)\Delta x \Delta y \Delta z \tag{6}$$

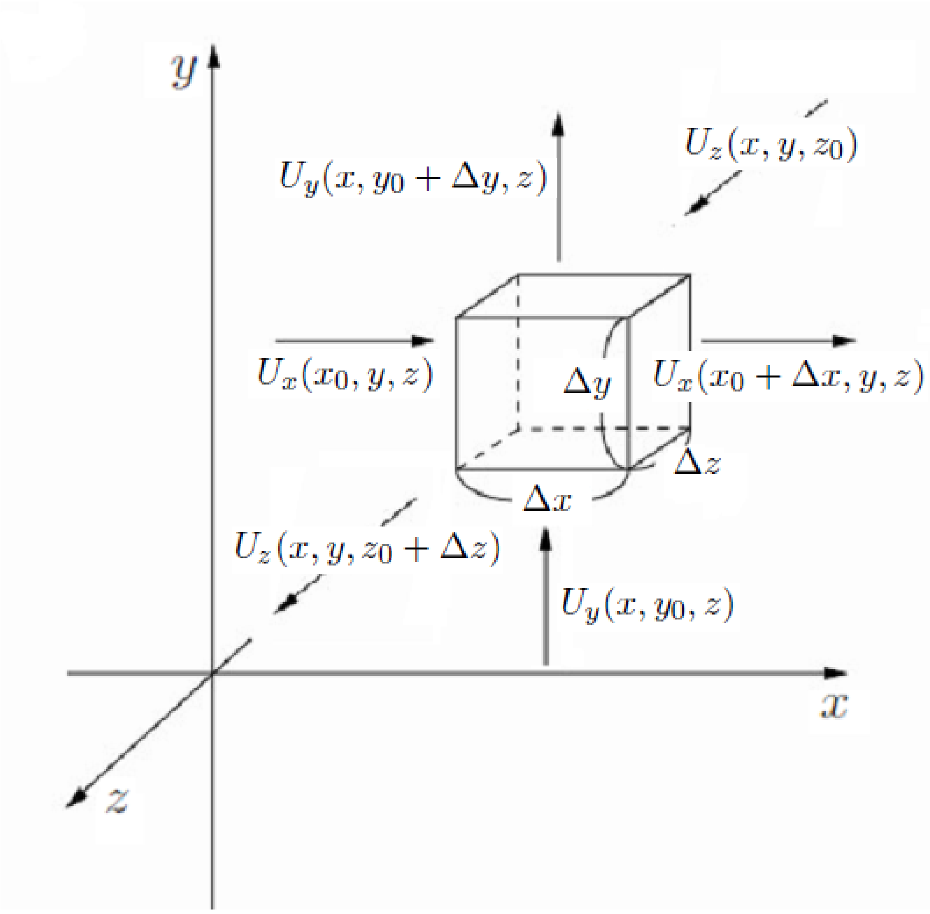


Figure 1: Divergence

And, similarly for U_z. In conclusion, the total flux coming out of the small region is given by:

$$\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right)\Delta x \Delta y \Delta z \tag{7}$$

Now, we have a enough motivation to define "divergence" as follows:

$$\text{div}\vec{U} \equiv \nabla \cdot \vec{U} \equiv \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} \tag{8}$$

If we denote the volume element as $dV = dxdydz$, then (7) becomes:

$$(\nabla \cdot \vec{U})dV \tag{9}$$

Given this, could we calculate the flux coming out of a certain, non-infinitesimal region in terms of divergence? In other words, could we express the following in terms of divergence?

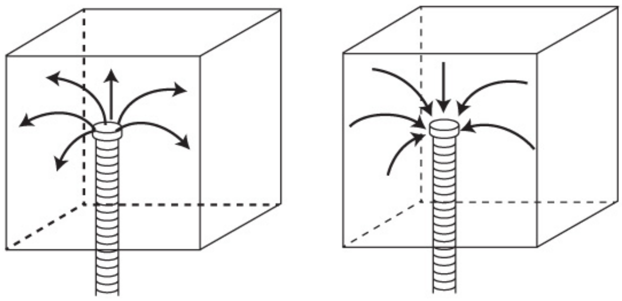
$$\oint_A \vec{U} \cdot dA \tag{10}$$

where A is a certain closed surface. (i.e. it has no boundary.) (The "o" in the integral sign means that A is a closed surface. Also, we expressed the double integral by a single integral for the convenience of notation. Remember that strictly speaking, it should be double integral because the area element $dA = dxdy$ needs two integration sign.) The answer is Yes. It is easy to see that the flux coming out of the surface A should be equal to the total sum of the flux coming out of the 3-dimensional volume bounded by A. Therefore (10) is equal to the volume integration of (9). Let's denote the bounded region as Ω, then the boundary of Ω is A. Mathematicians often denote the boundary by ∂. So, $A = \partial\Omega$. Therefore, we have proven the following equation, called "Stoke's theorem."

$$\int_\Omega \left(\nabla \cdot \vec{U}\right) dV = \oint_{\partial\Omega} \vec{U} \cdot dA \tag{11}$$

(Here, we expressed the triple integral in the left-hand side as a single integral for the convenience of notation.)

This may sound too abstract so let me explain this with an analogy. Suppose we have water flowing, and let's assume that the water is not compressible, i.e. the density is constant. Then, if water is literally created inside the cube in Fig. 1. the total flux out of the cube will be positive; the divergence of water's velocity at that point will be positive. If water is literally destroyed inside the cube, the total flux out of the cube will be negative; the divergence of water's velocity at that point will be negative. If water is neither generated nor destroyed inside the cube, but just passes by, there will be no net flux; the divergence will be zero. Of course, in reality, water is hard to be created or destroyed. The closest we can get is water coming out of a "source" or water sucking into a "sink." See the figure below.



On the left, water is being transported through a hose and is coming out of a source. In this case, the total flux coming out of the cube, if we ignore the flux going in through the hose, is positive. Thus, the divergence at the source (i.e. the tip of the hose) is positive. (Of course, if we ignore that the water is coming in through the hose.)

On the right, water is being sucked into the sink and transported out through a hose. In this case, the total flux coming out of the cube, if we ignore the flux going out through the hose, is negative. Thus, the divergence at the sink (i.e. the tip of the hose) is negative (Of course, if we ignore that the water is going out through the hose.)

In these examples, we considered a cube, but it can be any volume Ω as we saw in (11).

2 Continuity equation

We can re-express the continuity equation using Stoke's theorem as follows:

$$\begin{aligned} \int \frac{\partial \rho}{\partial t} dV &= -\oint (\rho \vec{v}) \cdot d\vec{A} \\ &= -\int \nabla \cdot (\rho \vec{v}) dV \end{aligned} \tag{12}$$

Therefore, we conclude:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{13}$$

In physics, we often denote $\vec{j} = \rho \vec{v}$, which yields:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \tag{14}$$

See Fig. 2 and Fig. 3. If the divergence of $\vec{j}$ is positive, it means that flux is going out of the cube, which means that the mass inside the cube

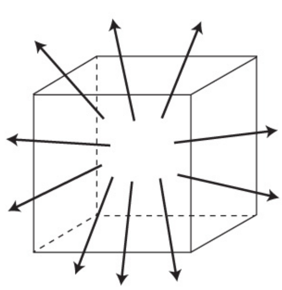


Figure 2: $\nabla \cdot \vec{j} > 0$

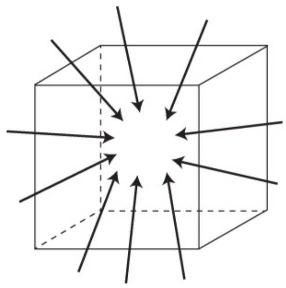


Figure 3: $\nabla \cdot \vec{j} < 0$

must be decreasing, as we have no other source of the mass. (We do not have a hose here which would supply the mass from outside.) Indeed, you can check from (14) that the density is decreasing in such a case. And, similarly, for the other case: if the divergence of $\vec{j}$ is negative, it means that flux is going into the cube, which means that the mass inside the cube must be increasing. (We do not have a hose here which would suck the mass and send it outside.) Indeed, you can check from (14) that the density is increasing in such a case.

Summary

- If there is a source, the divergence is positive. If there is a sink, the divergence is negative.
- Stoke's theorem is given by

$$\int_{\Omega} \left(\nabla \cdot \vec{U} \right) dV = \oint_{\partial\Omega} \vec{U} \cdot dA$$