The Doppler effect

In our earlier article "Doppler effects: How can we know the speed of the stars?" we approached the Doppler effect qualitatively. In this article, we will approach it quantitatively.

Let me recall what the Doppler effect is. When an ambulance approaches you, you hear the high pitch of its siren. However, after it has passed you and is moving away from you, the pitch suddenly drops. This is due to the Doppler effect. Let us explain why it is so.

Let's say that the siren of an ambulance at rest emits a sound with frequency f_0 and wavelength λ_0. If the speed of sound is given by V, we naturally have $V = \lambda_0 f_0$. Also, the period of the wave is given by $T = 1/f_0$. Now, suppose the ambulance is approaching a detector I and moving away from a detector J with speed v_A. See Fig.1. I have denoted the positions of the ambulance as A_1, A_2 and A_3, and the three wavefronts (i.e. circles) are emitted when the ambulance is at these positions. Given this, it is easy to imagine that detector I will meet the wavefronts more often than when the ambulance is not moving toward him. This is the reason why you hear the higher pitch of the siren. (The higher the pitch, the higher the frequency. For example, a pitch of an octave higher has double the frequency.) More specifically, while the wavefront comes to you at the speed of sound, the wavelength denoted here by λ_I is shorter than λ_0, which makes the pitch given by $f_I = V/\lambda_I$ higher than $f_0 = V/\lambda_0$. Similarly, detector J will meet the wavefronts less often than when the ambulance is not moving away from him. This is the reason why you hear the lower pitch of the siren. More specifically, while the wavefront comes to you at the speed of sound, the wavelength denoted here by λ_J is longer than λ_0, which makes the pitch given by $f_J = V/\lambda_J$ lower than $f_0 = V/\lambda_0$. Now, let's explicitly calculate these wavelengths. From the figure, we have:

$$VT_3 + v_A T + \lambda_I = VT_2, \qquad VT_3 - v_A T + \lambda_J = VT_2 \tag{1}$$

Using $T_2 = T_3 + T$, we get:

$$\lambda_I = (V - v_A)T, \qquad \lambda_J = (V + v_A)T \tag{2}$$

$$f_I = \frac{V}{\lambda_I} = f_0 \frac{V}{V - v_A}, \qquad f_J = \frac{V}{\lambda_J} = f_0 \frac{V}{V + v_A} \tag{3}$$

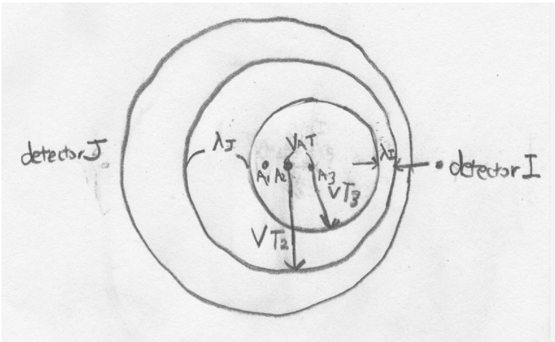

Figure 1: ambulance moving

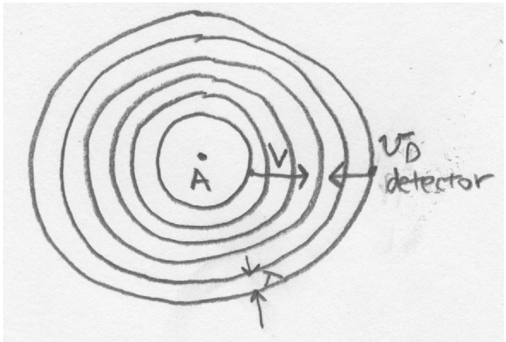

Figure 2: detector moving

This is the Doppler formula. Actually, we could have obtained the formula for λ_J and f_J by replacing v_A by $-v_A$ in λ_I and f_I since moving away with speed v_A is the same thing as approaching with speed $-v_A$.

Now, we move to the second case. This time, the ambulance is at rest and the detector is moving toward the ambulance at speed v_D. See Fig. 2. Again, it is easy to imagine that the pitch will be higher than when the detector is moving toward the ambulance, since it will meet the wavefronts more often. So, how often? During a time interval t, the wavefronts move toward the detector by the distance Vt while the detector moves toward the ambulance $v_D t$. So all together, the wavefronts and the detector get closer by $(V + v_D)t$ during time t. As you can see from the figure, the wavelength doesn't change, and during time t, the detector meets $(V+v_D)t/\lambda_0$ number of wavefronts. Therefore, for each second, the detector meets $(V+v_D)t/\lambda_0/t$ number of wavefronts. In conclusion, the detector detects the frequency of the siren to be

$$f_D = \frac{(V + v_D)t}{\lambda_0 t} = \frac{V + v_D}{V} f_0 \tag{4}$$

In the case that the detector is moving away from an ambulance with speed v_D, we can just replace v_D by $-v_D$ in the above equation.

Finally, what will be the detected frequency in case both the ambulance and the detector are moving? Combining (3) and (4), we get:

$$f = \frac{V \pm v_A}{V \mp v_D} f_0 \tag{5}$$

You can easily figure out when to use the plus sign or the minus sign.

Problem 1. Let's assume that v_A and v_D are much smaller than V. Then, show that(5) can be re-expressed as

$$\frac{f - f_0}{f_0} \approx \frac{V_r}{V} \tag{6}$$

where V_r is the relative velocity between the ambulance and the detector. In other words, if the relative velocity between the ambulance and the detector is 5% of the speed of sound, the observed frequency of the sound changes about 5%.

Summary

- If the relative velocity between the source and the detector is V_r and V_r is much smaller than V the speed of wave, the observed frequency change Δf is approximately given by

$$\frac{\Delta f}{f} \approx \frac{V_r}{V}$$

  where f is the original frequency.