The relativistic Doppler effect and the twin paradox revisited

In our earlier article "Doppler effect," we obtained that the observed frequency f is given by

$$f = \frac{V \pm v_A}{V \mp v_D} f_0 \tag{1}$$

where f_0 is the original frequency emitted by the source, and v_A is the velocity of the source, and v_D is the velocity of the detector.

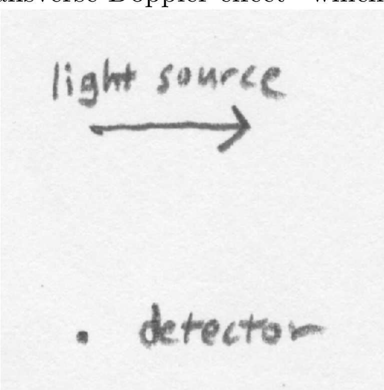
Notice that our treatment there was not relativistic. In Section 1 of this article, we will introduce its relativistic version. In Section 2, we will apply the Doppler effect to explicitly solve the twin paradox.

1 Relativistic Doppler effect

In this section, we will consider the Doppler effect for light waves. In particular, we will treat it relativistically.

First, notice that we cannot use (1) for this case. According to theory of special relativity, nature doesn't distinguish between who is moving, the detector or the light source. However, the formula (1) gives different results depending on who is moving.

In any case, let's find the correct formula. As the first step, we will introduce the "transverse Doppler effect" which has no non-relativistic analog. See the figure.



A light source is moving in a transverse direction to a detector, that is, it is neither approaching it nor moving away from it. Therefore, non-relativistically, there should be no Doppler effect. However, in relativity, there is time dilation. If the period of the light emitted from the source is given by t_0 in its rest frame, and the speed of the light source is given by v, the period that a detector will observe is simply given by:

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}} \tag{2}$$

Considering $f = 1/t$, and $f_0 = 1/t_0$, we conclude:

$$f = \sqrt{1 - v^2/c^2} f_0 \tag{3}$$

This is the formula for the transverse Doppler effect as advertised.

Now, let's consider the usual Doppler effect called the "longitudinal Doppler effect." In relativity, of course, the observer is not moving with respect to the observer. Therefore, if there is something that is moving, it should be the light source. We can therefore use (**??**), which is valid when the source is moving while the detector (i.e. observer) is not moving, provided that we plug in f in (3) for f_0 in (**??**). Therefore, if the light source is moving away from the detector with speed v, we have:

$$f = \frac{c}{c+v}\sqrt{1 - v^2/c^2} f_0 = \frac{\sqrt{1-v/c}\sqrt{1+v/c}}{1+v/c} f_0 = \sqrt{\frac{1-v/c}{1+v/c}} f_0 \tag{4}$$

Another way of seeing this is the following. From the point of view of the observer, the source travels a distance vt farther away between one tick. Therefore, between two consecutive ticks it takes vt/c seconds longer to reach the observer than the original interval of tick t. Therefore, we have:

$$f = \frac{1}{t + vt/c} = \frac{1}{t}\frac{c}{c+v} = \sqrt{1 - v^2/c^2} f_0 \frac{c}{c+v} = \sqrt{\frac{1-v/c}{1+v/c}} f_0 \tag{5}$$

In the case that the light source is moving toward the observer with speed v, we can plug in $-v$ for v, which results:

$$f = \sqrt{\frac{1+v/c}{1-v/c}} f_0 \tag{6}$$

2 Twin paradox

This section closely follows *Concepts of Modern Physics* by Arthur Beiser. Let's say that John, a twin brother of Michael's, departs to a distant star $L_0 = 20$ light years away with speed $v = 0.8c$ when they are both 20 years old, and returns to the earth with the same speed. Michael will remain on the earth. Michael will think that John takes $20/0.8 = 25$ years to get there, and 25 years to return. So, Michael will think that the total trip will take 50 years, and he will be 70 years old when John comes back. On the other hand, John will see that the distance to the distant star is shortened by the Lorentz contraction. The distance he will see is $L = L_0\sqrt{1 - v^2/c^2} = 20 \times 0.6 = 12$ light years. Therefore, from his point of view it takes $12/0.8 = 15$ years to get to the star and another 15 years to come back. John will be $20 + 30 = 50$ years old when he returns to the earth.

Now, let's say John and Michael exchange light signals every year. When John is moving toward the star, both will receive the light signal every

$$T_1 = t_0\sqrt{\frac{1 + v/c}{1 - v/c}} = 3 \text{ years} \tag{7}$$

while John is coming from the star, both will receive the light signal every

$$T_2 = t_0\sqrt{\frac{1 - v/c}{1 + v/c}} = \frac{1}{3} \text{ years} \tag{8}$$

From John's point of view, John receives 15 years/3 years=5 signals during his trip to the star, while he receives 15 years/(1/3) years=45 signals during his return trip. In total, he will receive 50 signals. Therefore, John will conclude that his brother is 20+50=70 years old, when he returns.

From Michael's point of view, John reaches the star after 25 years have elapsed. However, the signal John sent Michael when he arrived at the star takes an extra 20 years to reach Michael. Therefore, Michael will receive (25+20) years/3 years=15 signals that John sent Michael on the way to the star. So, Michael will indeed conclude that John aged 15 years on his way to the star. During John's return from the star, Michael will receive (25-20) years/(1/3) years=15 signals. So, Michael will indeed conclude again that John aged 15 years during his return trip. In other words, Michael will indeed think that John is 20+15+15=50 years old when he comes back to earth.

Summary

- Longitudinal Doppler effect can be derived from $t = \gamma t_0$. It is $f = f_0/\gamma$.
- Transverse Doppler effect is given by $f = \sqrt{\frac{c + v}{c - v}} f_0$.