The dot product

Consider the vectors $\vec{A} = A_x\hat{x} + A_y\hat{y}$ and $\vec{B} = B_x\hat{x} + B_y\hat{y}$ drawn in Figure 1. Then, is there a simple way to express θ, the angle between the two vectors in terms of their components A_x, A_y, B_x, and B_y? This is the question to which we will find an answer in this article.

First, if α is the angle $\vec{A}$ makes with the x-axis, we have:

$$\cos\alpha = \frac{A_x}{|\vec{A}|}, \qquad \sin\alpha = \frac{A_y}{|\vec{A}|} \tag{1}$$

where $|\vec{A}| = \sqrt{A_x^2 + A_y^2}$ is the magnitude of $\vec{A}$.

Similarly, if β is the angle $\vec{B}$ makes with the y-axis, we have:

$$\cos\beta = \frac{B_x}{|\vec{B}|}, \qquad \sin\beta = \frac{B_y}{|\vec{B}|} \tag{2}$$

where $|\vec{B}| = \sqrt{B_x^2 + B_y^2}$ is the magnitude of $\vec{B}$.

Given this, we now use the fact that $\theta = \beta - \alpha$. The trick is calculating its cosine as follows:

$$\begin{aligned} \cos\theta &= \cos(\beta-\alpha) = \cos\alpha\cos\beta + \sin\alpha\sin\beta && (3)\\ &= \frac{A_x}{|\vec{A}|}\frac{B_x}{|\vec{B}|} + \frac{A_y}{|\vec{A}|}\frac{B_y}{|\vec{B}|} && (4)\\ &= \frac{A_xB_x + A_yB_y}{|\vec{A}||\vec{B}|} && (5) \end{aligned}$$

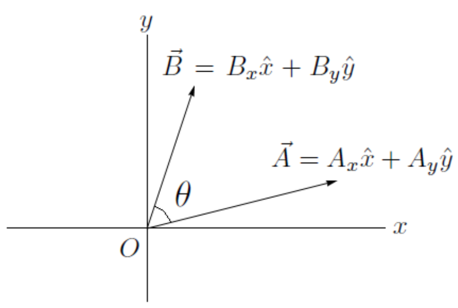

Figure 1: the angle θ between $\vec{A}$ and $\vec{B}$

Therefore, we found the answer to our question. θ is given by

$$\theta = \cos^{-1}\left(\frac{A_x B_x + A_y B_y}{|\vec{A}||\vec{B}|}\right) \tag{6}$$

This suggests us that our notation is simpler if we define the dot product between $\vec{A}$ and $\vec{B}$ as follows:

$$\vec{A} \cdot \vec{B} = (A_x \hat{x} + A_y \hat{y}) \cdot (B_x \hat{x} + B_y \hat{y}) = A_x B_x + A_y B_y \tag{7}$$

Then, we see that

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta \tag{8}$$

where θ is the angle between $\vec{A}$ and $\vec{B}$.

In higher-dimensions, the dot product is similarly defined. For example, in 3-dimensions, we have:

$$\vec{A} \cdot \vec{B} = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \cdot (B_x \hat{x} + B_y \hat{y} + B_z \hat{z}) = A_x B_x + A_y B_y + A_z B_z$$

In the next article, using "law of cosines," we will prove that (8) is satisfied in the 3-dimensional case as well, if we draw these two vectors in 3-dimensions.

Now, let me mention several properties of the dot product. First, the dot product is commutative. In other words, $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$. This follows from the fact that $A_x B_x + A_y B_y + A_z B_z = B_x A_x + B_y A_y + B_z A_z$. Second, the dot product is distributive. In other words, $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$. The check is left as an exercise to the readers. Third, the dot product of a vector with itself gives you the square of its magnitude. In other words, $\vec{A} \cdot \vec{A} = |\vec{A}|^2$. This is easily proven since $A_x A_x + A_y A_y + A_z A_z = A_x^2 + A_y^2 + A_z^2$.

We also would like to mention that the angle between two vectors is the right angle, if their dot product vanishes. This is evident from (8). If we plug in $\theta = 90°$, we get $\vec{A} \cdot \vec{B} = 0$. In such a case, we call $\vec{A}$ and $\vec{B}$ are orthogonal to each other.

Problem 1. What is the angle between two vectors $\vec{A} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{B} = -4\hat{i} - 4\hat{j} + 2\hat{k}$?

Problem 2. Check that $\cos\theta$ as obtained in (5) is necessarily between -1 and 1 using Cauchy-Schwarz inequality. Notice that this will be satisfied in higher-dimensions as well.

Summary

- The dot product of two vectors $\vec{A} = A_x \hat{x} + a_y \hat{y} + A_z \hat{z}$ and $\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$ is given by

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

- If the angle between $\vec{A}$ and $\vec{B}$ is θ, we have

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$$

- If two non-zero vectors $\vec{A}$ and $\vec{B}$ satisfy $\vec{A} \cdot \vec{B} = 0$, the two vectors are perpendicular to each other.
- $\vec{A} \cdot \vec{A} = |\vec{A}|^2$.
- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$.