Gauge transformation in dreibein

In this article, instead of vierbein, we will deal with dreibein, which will turn out to be useful in the analysis of Ashtekar variables. *Drei* means three in German. Dreibein is the same thing as vierbein, except for the dimension.

In the last article, we mentioned that dreibein (as well as vierbein) is not uniquely determined given a metric. In case of dreibein, we can still rotate it, and in case of vierbein we can Lorentz transform it. To see this, recall

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \tag{1}$$

If we choose another dreibein (or vierbein) that satisfies

$$e'^c_\mu = O^c_a e^a_\mu \tag{2}$$

where O^c_a is a rotation matrix (or Lorentz transformation matrix), it immediately follows from the definition of the rotation matrix (or Lorentz transformation matrix) that

$$g_{\mu\nu} = \eta_{cd} e'^c_\mu e'^d_\nu \tag{3}$$

Now, let's consider the infinitesimal version of (2). An infinitesimal rotation is given by a cross product. Thus, we can write

$$\delta \vec{e} = \vec{e} \times \vec{\Lambda} \tag{4}$$

In other words,

$$\delta_\Lambda e^a_\mu = \epsilon^a_{bc} e^b_\mu \Lambda^c \tag{5}$$

Now, let's find out how the spin connection transforms under this transform. To this end, it is convenient to introduce a new notation for the spin connection. Remember that a spin connection has two Lorentz-indices which are antisymmetric. For dreibein, 1, 2, 3 are possible for the Lorentz indices. Thus, there are three components for spin connections as 3 choose 2 is 3. Namely, we have

$$\omega^{32}_\mu = -\omega^{23}_\mu, \qquad \omega^{13}_\mu = -\omega^{31}_\mu, \qquad \omega^{21}_\mu = -\omega^{12}_\mu \tag{6}$$

Let's denote each of them by one Lorentz index by using Levi-Civita symbol as follows:

$$\omega^i_\mu = \frac{1}{2} \tilde{\epsilon}^i_{jk} \omega^{kj}_\mu \tag{7}$$

which implies

$$\omega_\mu^1 = \omega_\mu^{32} = -\omega_\mu^{23}, \qquad \omega_\mu^2 = \omega_\mu^{13} = -\omega_\mu^{31}, \qquad \omega_\mu^3 = \omega_\mu^{21} = -\omega_\mu^{12} \tag{8}$$

Now, we can calculate $\delta\omega_\mu^1 = \delta\omega_\mu^{32}$. From $\omega_{\mu ab} = e_a^\nu \nabla_\mu e_{b\nu}$, we have

$$\delta_\Lambda \omega_\mu^{32} = \delta e^{\nu 3} \nabla_\mu e_\nu^2 + e^{\nu 3} \nabla_\mu \delta e_\nu^2 \tag{9}$$

Using this, we obtain (**Problem 1.**)

$$\delta_\Lambda \omega_\mu^1 = \partial_\mu \Lambda^1 + \omega_\mu^2 \Lambda^3 - \omega_\mu^3 \Lambda^2 \tag{10}$$

Similarly, we get

$$\delta_\Lambda \omega_\mu^a = \partial_\mu \Lambda^a + \tilde{\epsilon}_{bc}^a \omega_\mu^b \Lambda^c \tag{11}$$

Given this, remember that a covariant derivative is given by

$$D\Lambda^b = d\Lambda^b + \omega^b{}_c \wedge \Lambda^c \tag{12}$$

For example,

$$\begin{aligned} D\Lambda^1 &= d\Lambda^1 + \omega^1{}_2 \wedge \Lambda^2 + \omega^1{}_3 \wedge \Lambda^3 & (13) \\ &= d\Lambda^1 + \omega^{12} \wedge \Lambda^2 + \omega^{13} \wedge \Lambda^3 & (14) \\ &= d\Lambda^1 + \omega^2 \wedge \Lambda^3 - \omega^3 \wedge \Lambda^2 & (15) \\ &= d\Lambda^1 + \tilde{\epsilon}_{bc}^1 \omega^b \Lambda^c & (16) \end{aligned}$$

Thus, we conclude

$$D\Lambda^a = d\Lambda^a + \tilde{\epsilon}_{bc}^a \omega^b \Lambda^c \tag{17}$$

Therefore, (11) can be re-written as

$$\delta_\Lambda \omega^a = D\Lambda^a \tag{18}$$

We will see this relation again when we talk about the non-Abelian gauge theory.

Problem 2. Check $R^{32} = d\omega^1 + \omega^2 \wedge \omega^3$.

Summary

- The vierbein has the Lorentz transformation as the gauge freedom and the dreibein has the rotation as the gauge freedom.
- $\delta\vec{e} = \vec{e} \times \Lambda$.
- $\delta_\Lambda \omega^a = D\Lambda^a$.