Is math and science homework mechanical drudgery?

It seems that at least some people think that math and science homework is mechanical drudgery devoid of creativity and imagination. They think that you just need to apply formulas, calculate things in prescribed ways, and get answers. In contrast, when you write a paper for your humanities class, you need to be creative and imaginative since there is neither a unique answer nor a prescribed way of solving the problems. They say that math and science homework is like cooking just by following a cookbook. You just cook dishes as prescribed by the cookbook.

They may be right, but only as far as THEIR math and science courses are concerned, i.e., low-level math and science courses such as those for high school students and college freshmen. I guess that I don't think that this cookbook method of solving problems works for higher-level courses or high school Math and Science Olympiad questions; you have to think hard and understand the problems before delving into them. You may, in most cases, need to tackle the problems in many different ways before finding out the right one that leads to the correct solution.

I was lucky enough to learn at an early age that math and science problem solving is not mechanical drudgery. I feel that it is somewhat unfortunate that those who claim that it is merely mechanical drudgery haven't had the chance to learn that it is not.

Let me give you an extreme example. A problem that is quite hard to solve with mechanical drudgery only, but a very easy problem if you find a creative way to solve it. See Fig. 1. There were seven bridges in Königsberg, Germany (present day Kaliningrad, Russia).

Starting from anywhere on the land or the islands (i.e., A, B, C, or D), is there any path in which you cross all the bridges without crossing the same one more than once? The answer is no. If you try to show this by brute force, it will take very long, which is indeed

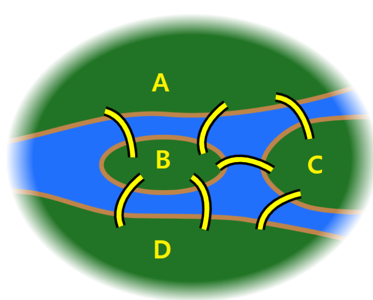

Figure 1: Seven bridges of Königsberg

a mechanical drudgery, because you have to exhaust all the possibilities. However, a Swiss mathematician Leonhard Euler proved that the answer was no by a very ingenious method in 1736. We will explain his proof at the end of the essay.

Japanese Fields Medalist Heisuke Hironaka wrote in "The Joy of Study" that he used to spend a lot of time solving difficult geometry problems when he was young. In the book, he showed one of them, which took him a week to figure out. Even though he provided a hint, I couldn't solve it. He recalled that times spent on solving such problems were helpful later when he performed real mathematical research.

Perhaps the way math and science are taught is wrong. We should encourage imagination in our teaching, if not creativity.

In 2009, I attended a talk about physics education by Professor Mazur at Harvard University. He noted that he used to think that he was a good teacher since he always received very high grades on the students' evaluations of his course.

However, he realized that this was a misperception. There is a multiple-choice type test called the FCI, which consists of about 30 questions that deal with Newtonian mechanics. This test focuses more on concepts than on problem-solving skills. He had his students take this test before and after they took his course and compared them. Surprisingly, he found out that there was not much difference in their performance before and after the course.

Then he said that the conventional physics problems checked only whether students are able to use physics formulas to plug in appropriate variables, rather than whether they actually understood the concepts.

As an example, he showed that students could easily solve electric circuit problems by using Kirchhoff's laws but failed to solve other electric circuit problems that focused more on concepts. The conceptual problems should be easier once students understand the concept since no algebra or formulas are needed. About half of the students got 0 or 2 points out of 10 on the conceptual problems.

Then, he presented his remedy to this problem; he proposed a new teaching method that can enhance students' understanding of concepts. If you are interested in this pedagogy, which he used with successful results, please check my journal entry on September 28th, 2009.

Anyhow, I think he is right. It's much more important to learn the concepts than to learn how to use physics formulas without thinking. I also think that standardized tests such as the SAT should ask such questions, even though it may turn out to be too hard to make enough problems.

Like homework, actual math and science research are not mechanical drudgery but requires tricks, new concepts, and critical thinking. In a long answer to an internet question, "What is it like to understand advanced mathematics? Does it feel analogous to having mastery of another language like in programming or linguistics?" "Anonymous" wrote, "To me, the biggest misconception that non-mathematicians have about how mathematicians work is that there is some mysterious mental faculty that is used to crack a research problem all at once." "Anonymous" also wrote, "In any case, by the time a problem gets to be a research problem, it's almost guaranteed that simple pattern matching won't finish it."

Let me comment here for those of you who learned the Pythagorean theorem in middle school or high school. Imagine you were Pythagoras. How would you find the relation between the sides in the right triangle? Is it as easy as your homework problem, if you didn't know the proof of the Pythagorean theorem beforehand? If you think so, I challenge you to come up with a proof of the Pythagorean theorem other than the ones you have already learned. It won't be as easy as you first imagined even though more than dozens of proofs of the Pythagorean theorem are already known. Indeed, there is no "mysterious mental faculty" that can be used to crack such a problem all at once. Even though you may know all the prerequisites to understand the proof of the Pythagorean theorem, to find one yourself is not easy. In our later articles, we will present four proofs of the Pythagorean theorem. While all these four proofs are not that difficult, in a genuine research problem, some proofs are easier or more elegant than the others of the same theorem. In our later articles "The center of mass of a triangle," and "The three altitudes of a triangle always meet at a point," I will present multiple approaches to the same problem that lead to the same solution. Then, you may be able to understand my statement that some proofs are easier or more elegant.

I agree with "anonymous"'s view. When I performed loop quantum gravity research, I accidentally found out a mysterious formula that fit our theoretical data. By theoretical data, I mean that we obtained them without performing actual physical experiments; my co-author wrote a computer code as prescribed by me, and he ran the code and obtained the data. At first, I had no idea why the mysterious formula held. As "anonymous" said, there was no "mysterious mental faculty that is used to crack" this problem. Then, four years later, I suddenly realized the first hint to the right solution and introduced a new concept. A year later, I performed another trick and gained more insight. Two years later, I found a "better" trick that supersedes the earlier trick. This example shows that "simple pattern matching" doesn't finish a research problem, but tricks, new concepts, and critical thinking are also needed.

In conclusion, it is unfortunate that some students erroneously think that math and science are mechanical drudgery because our way of teaching math and science is flawed. We should remedy this situation since real math and real science are not mechanical drudgery. Albert Einstein said, "Imagination is more important than knowledge." Math and science are not about knowledge, but about imagination, and we must be able to give such an impression to our students.

Euler's solution: First, Euler started off by simplifying the problem. See Fig. 2, which is a simplified picture of Fig. 1. As far as our problem is concerned, Fig. 2 contains all the information that Fig. 1 contains; Fig. 2 preserves the "connectedness" of Fig. 1. For example, in Fig. 1, there are two bridges between A and B, two bridges between B and D, one bridge between B and C, one bridge between A and C, one bridge between C and D. These are the essential information of Fig. 1, which Fig. 2 also contains. All the other

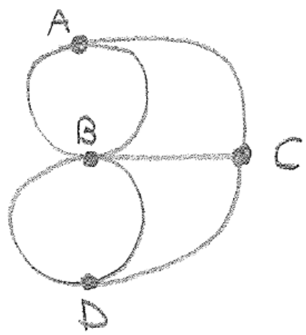

Figure 2: a simplifed picture of Fig. 1

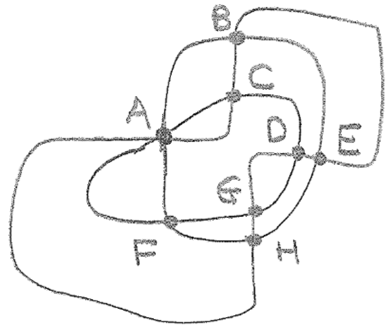

Figure 3: an Eulerian path

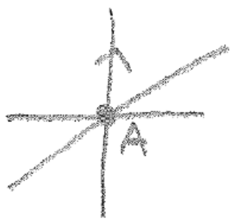

Figure 4: an Eulerian path starting from A

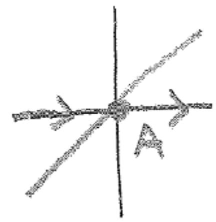

Figure 5: an Eulerian path passing A

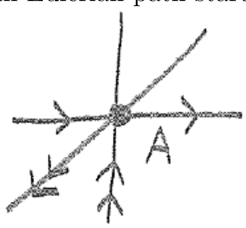

Figure 6: an Eulerian path passing A

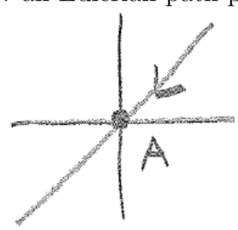

Figure 7: an Eulerian path ending at A

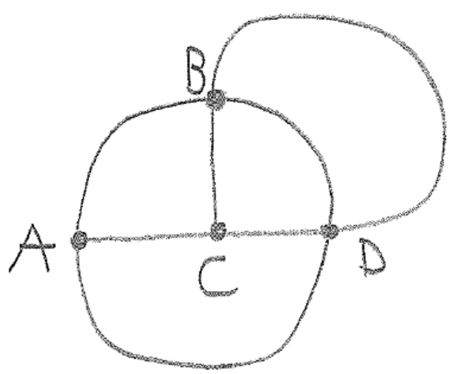

Figure 8: an Eulerian path

information is not important. For example, it doesn't matter at all, what shape the bridge between B and C has, or how long the bridges between A and B are, or the distance between the two bridges that connect B and D.

Given this, Euler studied what is now called "Eulerian path." A path that crosses its edge only once is called "Eulerian path." He deduced what properties an Eulerian path must have, and showed that "the seven bridges in Königsberg" does not satisfy this property.

Lets begin with an actual example of a diagram that has an Eulerian path. See Fig. 3. Suppose the Euler path starts at the node A, and ends at the node A. When the Euler path starts from A, it is "out-going" from A. See Fig. 4. I denoted this by an arrow on the edge. Then, the path will return to A, pass A and will leave A again. See Fig. 5. Every time it passes, it will have one "in-coming" edge to A and one "out-going" edge from A. In our case, it passes A total two times. See Fig. 6. Here, I denoted the second passing by double arrows. Finally, when it returns to A, it will have an "in-coming" edge to A. See Fig. 7. Total, there are 6 edges on A: 3 outgoing edges and 3 incoming edges. We easily see that there should be an equal number of outgoing edges and incoming edges at the node from which the Eulerian path starts and to which the Eulerian path ends at the same time. So, there should be an even number of edges at such a node. How about the other edges? The Eulerian path doesn't start or end there. It just passes. So, there must be also an equal number of outgoing edges and incoming edges for such a node. In other words, there must be an even number of edges for all the other edges. Summarizing, each node (both node such as A and nodes "passed") must have an even number of edges connected to it, if the Eulerian path returns to the same node from which it started.

Now, consider another Eulerian path. See Fig. 8. In this case, it starts and ends at different nodes. It starts at A and ends at C. A must have an outgoing edge when the Eulerian path starts. Then, when it returns to A to pass it, it must have an equal number of outgoing edges and incoming edges. In total, it must have one more outgoing edges than incoming edges, because it doesn't end at A, while it starts at A. This implies that the number of edges connected to A must be an odd number. (Say the number of incoming edges at A is n. Then the number of outgoing edges at A is $n + 1$. If you add $n + 1$ and n, it is $2n + 1$, which is always an odd number.) Similarly, at C, there should be one more incoming edges than the outgoing edges, because the Eulerian path didn't start at C, while it ends there. Therefore, C must have an odd number of edges. At all the other points, the Eulerian path just passes it. Thus, they have an equal number of outgoing edges and incoming edges. Therefore, each of them must have an even number of edges. Summarizing, if an Eulerian path starts from a node and returns to another node, only these two nodes must have an odd number of edges, and all the other nodes must have an even number of edges.

Now, see Fig. 2, the simplified picture of Fig. 1. Given this notice that each node (A, B, C, D) has an odd number of edges (i.e., bridges) connected to it. In other words, four nodes have an odd number of edges in this case. As a diagram must have only zero or two

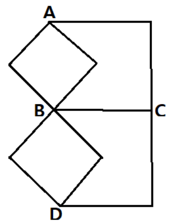

Figure 9: another representation of Fig. 2

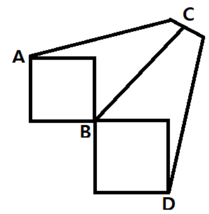

Figure 10: another representation of Fig. 2

nodes that have an odd number edges, there is no Eulerian path for the seven bridges of Königsberg.

I learned the condition for a diagram to have the Eulerian path in special classes for gifted students in elementary school. However, I just memorized the condition without learning the derivation, which is the fun part. I had to prepare for competitions, which never asked such derivations, but only whether a particular diagram had an Eulerian path. If I could have taught young myself, I would have done it differently.

A final comment. In the problem of bridges of Königsberg, we have seen that not the exact shape of the bridges and the island but only their connectedness mattered. For example, Fig. 9 and Fig. 10 can be other equivalent representations of Fig. 1 or Fig. 2 as far as the connectedness is concerned. Euler's solution to this problem gave rise to a new important concept in mathematics: Topology. In topology, all that matters is not the exact shape, but the connectedness. Fig. 2, Fig. 9, and Fig. 10 have all the same connectedness. Notice that to preserve connectedness, you can stretch, twist, bend the lines, but you must not glue or cut the lines. If the connectedness is preserved during a transformation, we say that the transformation preserves "topology." We can also say that Fig. 2, Fig.9 and Fig. 10 have the same "topology."

Topology is now a very hugh subject in mathematics. In this example, you see that what we need to create a new field or subject in mathematics is not boring logical deductions, but an ingenious way of looking at the same problem. We will talk more about topology in our article "Topolgy, the Euler characteristic, and the Gauss-Bonnet theorem." There, you will also see an example of how different subjects in mathematics connect each other, unexpectedly.

Problem 1. Suppose that a diagram admits an Eulerian path that starts from a certain node which it can end (i.e., all nodes have even edges.) Then, show that such a diagram admits an Eulerian path that can start from *any* nodes.

(Fig. 1 is adopted from `https://commons.wikimedia.org/wiki/File:7_bridges.svg`.)