Dual space

Let V be a real vector space. Then, consider the following linear map (i.e., linear operator) L.

$$L : V \to \mathbb{R} \tag{1}$$

In other words, if you give the linear map a vector in a vector space V, the linear map gives you a real number. The set of all such linear maps is called "dual space" and is denoted by V^*. Let's find this dual space. To this end, recall what a linear map is from our earlier article "Matrices and Linear Algebra." It satisfies the following two conditions:

$$L(c\vec{v}) = cL(\vec{v}), \quad L(\vec{v}_1 + \vec{v}_2) = L(\vec{v}_1) + L(\vec{v}_2) \tag{2}$$

In that article, we have learned that a linear map can be always represented by a matrix. Let's say that V is n-dimensional. Then, our linear map L pops out one number when n numbers (recall that a n-dimensional vector can be uniquely represented by n numbers) are entered. So, we see that L must be $1 \times n$ matrix. We know that $n \times 1$ matrix forms an n-dimensional vector space. In other words, if V is an n-dimensional vector space, its dual space V^* is also an n-dimensional vector space.

Maybe, you should think along this way. If V is a ket vector (i.e., $n \times 1$ matrix), V^* is a bra vector (i.e., $1 \times n$ matrix). The real number is obtained by their dot-product. Of course, it goes without saying that the dual space of V^* is the original vector space V. In other words, $(V^*)^* = V$.

A comment. In our later article on general relativity, you will learn that a usual vector (i.e., ket vector) is called just a vector, and a vector that lives in the dual vector space (i.e., bra vector) is called a "covector." Actually, we have already encountered them when we learned Einstein summation convention. An object with upper index such as B^i is a vector and an object with lower index such as A_i is a covector. They combine together ($A_i B^i$) to give out a number. This is just another expression of dot product.

All I said in this article can be generalized to the case in which V is a complex vector space. The only difference is that, instead of (1), we have the following:

$$L : V \to \mathbb{C} \tag{3}$$

Summary

- The set of all linear maps which give you back a number when you give them a vector is called “dual space.”
- The dual space of V is denoted by V^*. If V is n-dimensional, V^* is n-dimensional.
- The pairing of V and V^* is exactly what a dot product is.
- $(V^*)^* = V$
- If a vector lives in a vector space, a covector lives in its dual space.
- A vector has an upper index as v^i and a covector has a lower index as u_i. A vector and a covector can be combined together to yield a dot product.