Eigenvalues and eigenvectors of symmetric matrices and Hermitian matrices

In this article, we will learn very interesting properties of eigenvalues and eigenvectors of symmetric matrices. To this end, let's recall what eigenvalues and eigenvectors are. Given A an $n \times n$ matrix, it is λ_i (a number) and e_i (an $n \times 1$ matrix) that satisfy the following equation.

$$Ae_i = \lambda_i e_i \tag{1}$$

where i runs from 1 to n. Let me give you an example. If A is given by following:

$$A = \begin{pmatrix} -2 & 4 & -1 \\ -1 & 3 & -1 \\ 2 & -2 & 1 \end{pmatrix} \tag{2}$$

we find the following solutions (I will briefly mention how one can find the solutions in my article "Finding eigenvalues and eigenvectors."):

$$\begin{pmatrix} -2 & 4 & -1 \\ -1 & 3 & -1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3}$$

which corresponds to

$$Ae_1 = 1e_1 \tag{4}$$

$$\begin{pmatrix} -2 & 4 & -1 \\ -1 & 3 & -1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{5}$$

which corresponds to

$$Ae_2 = 2e_2 \tag{6}$$

$$\begin{pmatrix} -2 & 4 & -1 \\ -1 & 3 & -1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{7}$$

which corresponds to

$$Ae_3 = -1e_3 \tag{8}$$

We can express the same equations slightly differently. Let's take the transpose of (1). Then we have:

$$\begin{aligned}(Ae_i)^T &= (\lambda_i e_i)^T \\ (e_i)^T A^T &= \lambda_i (e_i)^T\end{aligned} \tag{9}$$

For example, (5) becomes:

$$\begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & -1 & 2 \\ 4 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \tag{10}$$

Now, assume a matrix B is a symmetric matrix, (i.e. $B^T = B$) and let's say that v_is are eigenvectors and λ_is are eigenvalues.

Then, from (1), we have

$$Bv_i = \lambda_i v_i \tag{11}$$

On the other hand, from (9), we have:

$$(v_i)^T B^T = \lambda_i (v_i)^T \tag{12}$$

However, as $B^T = B$, we have

$$(v_i)^T B = \lambda_i (v_i)^T \tag{13}$$

Now consider the following quantity:

$$(v_j)^T B v_i = (v_j)^T \lambda_i v_i = \lambda_i (v_j)^T v_i \tag{14}$$

which follows from (11) On the other hand, we also have

$$(v_j)^T B v_i = \lambda_j (v_j)^T v_i \tag{15}$$

which follows from (13)

Equating the above two, we conclude:

$$\begin{aligned}\lambda_i (v_j)^T v_i &= \lambda_j (v_j)^T v_i \\ (\lambda_i - \lambda_j)(v_j)^T v_i &= 0\end{aligned} \tag{16}$$

So, if $\lambda_i \neq \lambda_j$, we have:

$$(v_j)^T v_i = 0 \tag{17}$$

However, notice that $(v_j)^T v_i = \vec{v}_j \cdot \vec{v}_i$ from the definition of dot product. For example, if

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \tag{18}$$

then it follows that

$$(v_1)^T v_2 = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = 1 \times 4 + 2 \times 0 + (-1) \times 3 = \vec{v}_1 \cdot \vec{v}_2 \tag{19}$$

Therefore, from (17), we have:

$$\vec{v}_j \cdot \vec{v}_i = 0 \tag{20}$$

In other words, two eigenvectors of a symmetric matrix are orthogonal to each other, if their eigenvalues are different.

Everything presented here can be translated into the case in which the symmetric matrix is replaced by Hermitian matrix. (9) is replaced by following equations.

$$\begin{aligned} (Ae_i)^\dagger &= (\lambda_i e_i)^\dagger \\ (e_i)^\dagger A^\dagger &= \lambda_i^* (e_i)^\dagger \end{aligned} \tag{21}$$

Taking similar steps to the ones that led to (16), we obtain:

$$\begin{aligned} \lambda_i (v_j)^\dagger v_i &= \lambda_j^* (v_j)^\dagger v_i \\ (\lambda_i - \lambda_j^*)(v_j)^\dagger v_i &= 0 \end{aligned} \tag{22}$$

Now, consider the case $j = i$. Then we see that $\lambda_i = \lambda_i^*$ which means that λ_i is real. So, we have shown that eigenvalues of Hermitian matrices are always real.

Considering this, we can write the above formula as

$$(\lambda_i - \lambda_j)(v_j)^\dagger v_i = 0 \tag{23}$$

If $\lambda_i \neq \lambda_j$. we have:

$$(v_j)^\dagger v_i = 0 \tag{24}$$

If we define the inner product (also known as dot product) of two complex vectors as follows:

$$\vec{v}_j \cdot \vec{v}_i = (v_j)^\dagger v_i \tag{25}$$

We conclude that two eigenvectors of a Hermitian matrix are orthogonal to each other, if their eigenvalues are different. Notice that the definition of dot product in complex vector case is slightly different from the real one. For example, if you had

$$v_1 = \begin{pmatrix} 1-i \\ 2 \\ -1+i \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ 0 \\ 3-2i \end{pmatrix} \tag{26}$$

$$\begin{aligned} \vec{v}_1 \cdot \vec{v}_2 &\equiv (v_1)^\dagger v_2 = \begin{pmatrix} 1+i, & 2, & -1-i \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 3-2i \end{pmatrix} \\ &= (1+i) \times 4 + 2 \times 0 + (-1-i) \times (3-2i) \end{aligned} \tag{27}$$

Notice that the dot product is not given by $(1-i)\times 4+2\times 0+(-1+i)\times(3-2i)$. We will revisit this issue in my fourth article on quantum mechanics. There, we will also see that the orthogonality of eigenvectors of Hermitian matrix has a very far-reaching significance in quantum mechanics. Additionally, we will express everything in this article in terms of bra-ket notation, introduced in my earlier article. Readers could have left out this article, and read my fourth article on quantum mechanics, and still understand it, but it is good to learn twice.

Problem 1. If A is a Hermitian 2×2 matrix with two distinct eigenvalues, and one of its eigenvector u_1 is given by the following:

$$u_1 = \begin{pmatrix} i \\ 2 \end{pmatrix} \tag{28}$$

Show that the other eigenvector is given by,

$$u_2 = \begin{pmatrix} 2 \\ i \end{pmatrix} \tag{29}$$

Problem 2. If B is a Hermitian 3×3 matrix with three distinct eigenvalues, and two of its eigenvectors v_1 and v_2 are given by the following:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ i \\ 3 \end{pmatrix} \tag{30}$$

What is the third eigenvector v_3? (The answer is not unique, so just give me an example of possible answers.[1])

[1]However, such eigenvectors are always parallel (or anti-parallel) to each other. In other words, two such eigenvectors $\vec{v}_3$ and $\vec{v}_3'$ always satisfy $\vec{v}_3' = c\vec{v}_3$ for a suitable scalar c.

Summary

- The eigenvectors of a Hermitian matrix are orthogonal to each other if their eigenvalues are different.