Einstein summation convention

Einstein summation convention was invented by Albert Einstein. It is a convention that simplifies the notation of certain types of sums that are used frequently in Einstein's theory of general relativity and particle physics. It says that if an index appears both as an upper index and as a lower index, a sum is calculated as the index runs over from 1 to n, even if the summation is not explicitly denoted. You can easily denote matrix calculation or tensor calculation by using this notation.

Now, I will explain how you can denote a scalar product (also known as dot product or inner product) by using this notation.

A scalar product between three-dimensional vectors $\vec{A}$ and $\vec{B}$ can be written as $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$. If you consider the first component as x, the second component as y and the third component as z, you may rewrite this summation as $\vec{A} \cdot \vec{B} = A^1 B_1 + A^2 B_2 + A^3 B_3$. If you write this more succinctly, you can write this as $\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A^i B_i$.

If you use Einstein summation convention, then you can write this as $\vec{A} \cdot \vec{B} = A^i B_i$. Because i appears both as an upper index and as a lower index, it is implied that a sum is calculated as i runs over the integers from 1 to n. Such a variable i which appears both as an upper index and as a lower index is called "dummy variable" or "dummy index." You can always change dummy variable to another letter. For example: $A^i B_i = A^j B_j$

Let's also denote a matrix multiplication by Einstein summation convention. Let $C = AB$, where A,B and C are matrices. If you write this out in components you get $C^i_k = \sum_{j=1}^{n} A^i_j B^j_k$. If you use Einstein summation convention, you get $C^i_k = A^i_j B^j_k$.

Also, there is no reason why there should be only one dummy variable. For example, let's write the relation between Ricci scalar and Ricci tensor by using Einstein summation convention. (Ricci scalar and Ricci tensor are central objects in Einstein's theory of general relativity.) The relation can be written as

$$R = \sum_{a=1}^{4} \sum_{b=1}^{4} g^{ab} R_{ab} = g^{ab} R_{ab}$$

As you see, there can be multiple dummy variables. In that case, the sum is calculated by all dummy variables running over all the integers from 1 to n.

Problem 1. Show the following.

$$A_{ab}B^{ab} + C_{cd}B^{cd} = (A_{ab} + C_{ab})B^{ab}$$

Problem 2. Let $A^{ab} = A^{ba}$ and $B_{ab} = -B_{ba}$. (Such objects are called "symmetric rank 2 tensors" and "anti-symmetric rank 2 tensors," respectively. Here, 2 denotes the number of indices.) Prove (Hint[1])

$$A^{ab}B_{ab} = 0$$

Summary

- If an index appears both as an upper index and as a lower index, a summation on that index is understood. Such an index is called "dummy variables."

[1] Use change of the dummy variables.