Electrodynamics in the Lagrangian and the Hamiltonian formulations

What is the Lagrangian and the Hamiltonian of a charged particle in the presence of the electric field, but no magnetic field? It's easy. If the electric potential is ϕ, the potential energy, V is given by $q\phi$ where q is the charge of the particle. Therefore, we have:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\phi \tag{1}$$

$$H = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + q\phi \tag{2}$$

In the presence of the magnetic field, it is more complicated. We will guess a formula for the Lagrangian and show that the correct equation of motion is derived from the Lagrangian, thus justifying the formula. We guess:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + q(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) - q\phi \tag{3}$$

Then, we have:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x \tag{4}$$

and similarly for p_y and p_z. Then, we have:

$$\dot{p}_x = m\ddot{x} + q\frac{d}{dt}A_x = m\ddot{x} + q\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right) \tag{5}$$

$$= \frac{\partial L}{\partial x} = q\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}\right) - q\frac{\partial \phi}{\partial x} \tag{6}$$

Therefore, we conclude:

$$m\ddot{x} = q\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\dot{y} + q\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\dot{z} - q\frac{\partial \phi}{\partial x} \tag{7}$$

However, remembering that $B = \nabla \times A$ and $E = -\nabla\phi$ we conclude:

$$m\ddot{x} = qE_x + q(B_z\dot{y} - B_y\dot{z}) \tag{8}$$

and similarly for $m\ddot{y}$ and $m\ddot{z}$. In conclusion, we obtained the Lorentz force as follows:

$$m\ddot{\vec{r}} = q\vec{E} + q\dot{\vec{r}} \times \vec{B} \tag{9}$$

where $\vec{r} = (x, y, z)$.

Now, by definition, Hamiltonian is given by:

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L \\ &= p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + q(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) - q\phi\right) \end{aligned} \tag{10}$$

Now, (4) implies:

$$\dot{x} = \frac{p_x - qA_x}{m} \tag{11}$$

and similarly for $\dot{y}$ and $\dot{z}$. Plugging this into (10), we get:

$$H = \frac{1}{2m}\left((p_x - qA_x)^2 + (p_y - qA_y)^2 + (p_z - qA_z)^2\right) + q\phi \tag{12}$$

which is equal to:

$$H = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + q\phi \tag{13}$$

which is the total energy (i.e. kinetic energy plus potential energy) of moving charged object. Notice that p's in (12) is not the "mechanical" momentum given by $m\dot{\vec{r}}$. From (11), it is easy to see that mechanical momentum is given by $\vec{p} - q\vec{A}$.

Summary

- The Hamiltonian of charged object in a vector potential $\vec{A}$ and an electric potential ϕ is given by

  $$H = \frac{1}{2m}\left((p_x - qA_x)^2 + (p_y - qA_y)^2 + (p_z - qA_z)^2\right) + q\phi$$

  Here, $\vec{p} - q\vec{A}$ is the mechanical momentum.