Electromagnetic duality

In this article, we will delve into the electromagnetic duality that we briefly mentioned in "Light as electromagnetic waves." There, we found that the transformation

$$\vec{E} \to \vec{B}, \qquad \vec{B} \to -\vec{E} \tag{1}$$

preserves the Maxwell equations in vacuum. Then, from the explicit form of F given in the last article, it is easy to check the above duality transformation can be re-written as

$$F \to *F \tag{2}$$

(**Problem 1.** Check that $**F = -F$.) However, the Maxwell's equations in the presence of electric charges are not invariant under this transformation, because

$$dF = 0 \to d*F = 0, \qquad d*F = J \to -dF = J \tag{3}$$

To remedy this situation, we need to introduce J^m, the magnetic 4-current, and write the Maxwell's equations as

$$dF = J^m, \quad d*F = J \tag{4}$$

and introduce the additional duality transformation besides (2)

$$J \to -J^m, \qquad J^m \to J \tag{5}$$

Problem 2. Let's write the magnetic 4-current as follows

$$J^m = (-j_x^m dy \wedge dz - j_y^m dz \wedge dx - j_z^m dx \wedge dy) \wedge dt + \rho^m dx \wedge dy \wedge dz \tag{6}$$

Then, derive the new Maxwell's equations expressed in terms of curl, divergence, and the partial derivative with respect to t. If you got $\mathrm{div} B = \rho^m$, as one of the two new equations, you are correct. This is Gauss's law for magnetic field in the presence of magnetic monopole, i.e. magnetic charge. The other equation should involve the magnetic current.

Problem 3. In these new Maxwell's equations (4) in the presence of magnetic charge, can we still express the electromagnetic field using one form A as $F = dA$? Why or why not?

Problem 4. Do these new Maxwell's equations imply the conservation of magnetic charge?

In our later article "Dirac string," we will explain why the electromagnetic duality is S-duality.

Summary

- In the presence of magnetic charge (magnetic monopole), Maxwell's equations become $dF = J_m,\ d*F = J$.