Ellipse revisited

As promised in our earlier article "Conic sections," we re-visit ellipse in this article.

1 Area of an ellipse

Remember that an ellipse is a "squeezed" or "re-scaled" or "stretched" circle, according to one of our earlier several equivalent definitions of an ellipse. See Fig. 1 and Fig. 2. Fig. 1 is a circle. Its radius is "10," if we call the spacing between adjacent pink lattice "1." In Fig. 2, we rescaled Fig. 1 by stretching the x-coordinate by the ratio 1.4 while leaving the y-coordinate un-changed. Therefore, the semi-major axis a of the ellipse is 14, while its semi-minor axis b is 10.

Then, what is the formula for the area of an ellipse with semi-major axis a and semi-minor axis b? Remember that the area of a circle with radius r is given by πr^2. In Fig. 1, the number of squares inside the circle is the area of the circle. I didn't count it, but it should be around 314, as $r = 10$. (Of course, we can get a more exact value by making the lattice denser). In Fig. 2, the number of rectangles inside the ellipse remains the same, because we just stretched Fig. 1; the lattices were stretched as well. However, the area of each rectangle is now 1.4 ($= 1.4 \times 1$). Therefore, the area of the ellipse must be around 314×1.4.

See what we just have done. The area of this ellipse is given by

$$A = \pi \times 10 \times 10 \times 1.4 = \pi \times 10 \times 14 = \pi ba \tag{1}$$

In other words, remembering that $b = r$, and $1.4 = a/r$,

$$A = \pi \times r \times r \times \frac{a}{r} = \pi \times b \times a = \pi ba \tag{2}$$

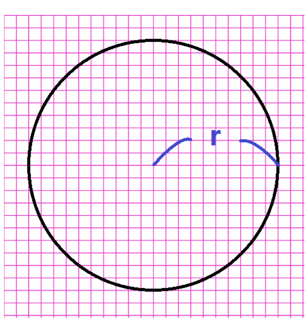

Figure 1: a circle

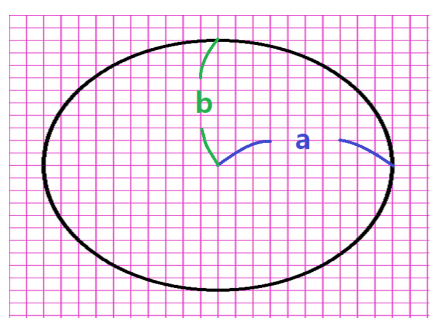

Figure 2: an ellipse

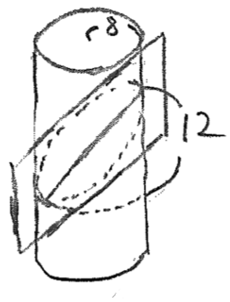

Figure 3: Problem 1.

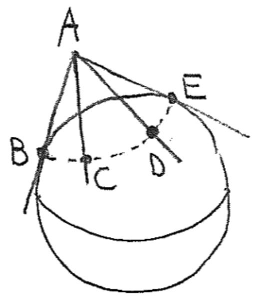

Figure 4: tangents from A

Problem 1. Using a knife, you sliced a cylinder with radius 8 as in Fig. 3. You can also see that the vertical displacement in the slicing along the cylinder is 12. If your knife is not bent, the cross section is an ellipse, because the circle with radius 8 is stretched by a constant ratio along one direction this way. What is the area of this ellipse? (Hint[1])

2 Two foci and ellipse as a stretched circle

In an earlier article, we mentioned that ellipse can be defined by the condition that the sum of the distances from a point on an ellipse to the two foci is constant. Now, we will show that a stretched circle, as a section of a cylinder, satisfies this property. Before doing so, we need to prove a theorem. See Fig. 4. You see a sphere and a point A. You also see four tangents from A to the sphere. These four tangents touch the sphere at B, C, D, E. What we want to say is that all the tangents from A to the sphere have the same length. In other words,

$$\overline{AB} = \overline{AC} = \overline{AD} = \overline{AE} \tag{3}$$

Of course, this is true for other tangents from A, which we drew not here. But they will touch the points denoted by the dotted line.

Problem 2. Let's say that the radius of the sphere is r and the distance from the center of the sphere to A is a. Then, calculate the value for $\overline{AE}$. (Hint[2]) Note that your calculation is valid for all the other tangents. In other words, you will get the same answer for $\overline{AB}$, $\overline{AC}$, $\overline{AD}$. So, you just proved our assertion. They are the same.

Given this, see Fig. 5. You see a cylinder with radius r and height h. In this cylinder, two spheres S_1, S_2, both with radius r are tightly fit. You also see a section of the cylinder. It touches the lower sphere at the point F_1, and the upper sphere at the point F_2. In other words, this section is a tangent plane of both spheres.

Now, recall that this section is an ellipse, because it is a "stretched" circle. What I am going to show is that the two points F_1 and F_2 are actually foci. See Fig. 6. All we need to

[1]Use the Pythagorean theorem to calculate the semi-major axis.
[2]Use the Pythagorean theorem.

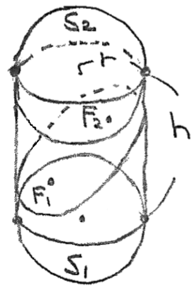

Figure 5: a cylinder with two spheres

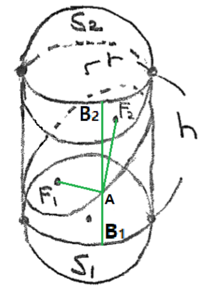

Figure 6: $\overline{AF_1} + \overline{AF_2}$

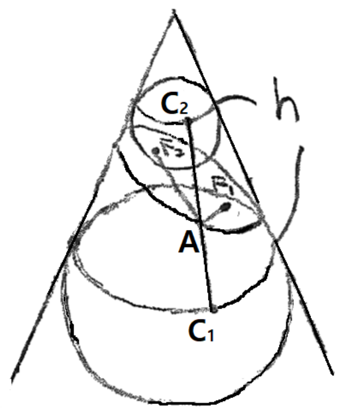

Figure 7: Ellipse as a conic section

prove is $\overline{AF_1} + \overline{AF_2}$ is constant no matter where we choose A as long as it is on the ellipse. Now, notice $\overline{AF_1} = \overline{AB_1}$, because both $\overline{AF_1}$ and $\overline{AB_1}$ are tangents to the sphere S_1 from the same point A. Similarly, $\overline{AF_2} = \overline{AB_2}$, as they are tangents to the sphere S_2 from the same point A. Thus, we conclude

$$\overline{AF_1} + \overline{AF_2} = \overline{AB_1} + \overline{AB_2} = h \tag{4}$$

In other words, the sum of the distances from a point on an ellipse to the foci is always a constant. This completes the proof.

3 Ellipse as a conic section

In this section, you will be invited to prove that an ellipse, as a section of a conic, satisfies the condition that the sum of the distances to the two foci is always a constant.

Problem 3. See Fig. 7. The figure is self-explanatory. Explain why $\overline{AF_1} = \overline{AC_1}$ and $\overline{AF_2} = \overline{AC_2}$. Then, explain why

$$\overline{AF_1} + \overline{AF_2} = h \tag{5}$$

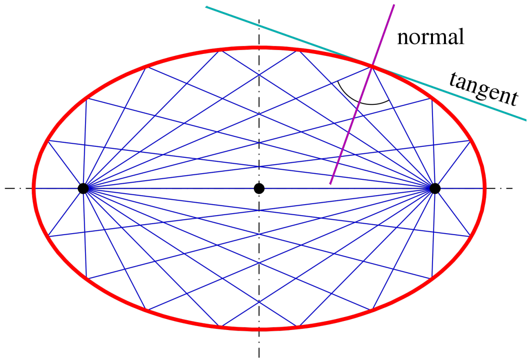

Figure 8: an elliptical mirror and two foci

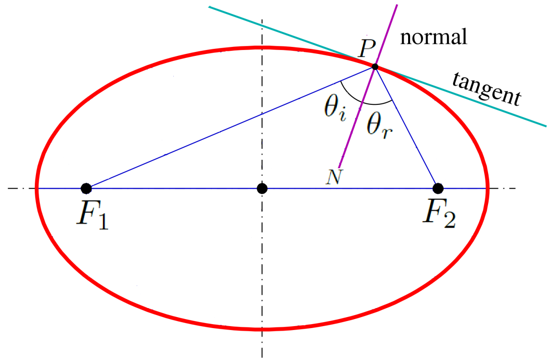

Figure 9: θ_i and θ_r

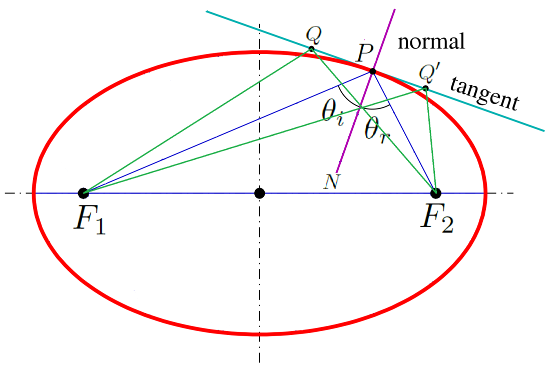

Figure 10: the shortest path is the one through P

4 An elliptical mirror

Imagine an elliptical mirror as in Fig. 8. If you shoot light rays from one of the focus of the ellipse, they will bounce back from the elliptical mirror, and converge at the other focus. I will explain the reason why in this section.

To explain this, we need to use the fact that the incident angle and the reflected angle are the same on a mirror, which we proved in our earlier essay “Light ray reflecting on a mirror.” See Fig. 9. If we shoot the light ray from F_1 to P, the incident angle is $\theta_i = \angle F_1PN$. In the figure, we also drew $\theta_r = \angle NPF_2$. If θ_r happens to be equal to θ_i, then a light ray shot from F_1 toward P will bounce off at P and be reflected to F_2. Thus, all we need to prove is $\theta_r = \theta_i$.

Given this, see Fig. 10. If the semi-major axis is a, we have

$$\overline{F_1P} + \overline{F_2P} = 2a \tag{6}$$

Let’s now call two arbitrary points on the tangent that touches P by Q and Q'. As Q and Q' lie outside of the ellipse, we necessarily have

$$\overline{F_1Q} + \overline{F_2Q} > 2a, \qquad \overline{F_1Q'} + \overline{F_2Q'} > 2a \tag{7}$$

unless they coincide with P.

In other words, as we move Q (or Q') along the tangent, $\overline{F_1Q} + \overline{F_2Q}$ is minimum, when Q is located at P, in which case the value is $2a$. Given this, remember from our earlier essay on Fermat's principle that such P necessarily satisfies $\angle F_1PN = \angle F_2PN$. This completes the proof.

Final comment. One can prove that a hyperbola defined by a conic section satisfies the condition that the difference between the distances to the foci is always constant, using a similar method to the geometric one which we used to prove that an ellipse defined as a conic section satisfies the condition that the sum of the distances to the foci is always constant. Another similar geometric method can show that a parabola is indeed a conic section. However, we will not show the proofs. Interested readers can consult other books.

Summary

- The area of an ellipse with semi-major axis a and semi-minor axis b is given b πab.
- If you shoot light rays from one of the foci of an elliptical mirror, they all converge to the other focus.

(Fig. 8 is from `https://commons.wikimedia.org/wiki/File:Elli-norm-tang-n.svg`)