Energy density of electromagnetic field

1 The energy density of electric field

If there are n objects with charge q_1, q_2, $\cdots$, q_n at positions $\vec{x}_1$, $\vec{x}_2$, $\cdots$, $\vec{x}_n$, what is the electric potential at the point where the charge q_i is? (i.e. at position $\vec{x}_i$.) It is:

$$\Phi(\vec{x}_i) = k \sum_{j=1, j\neq i}^{n} \frac{q_j}{|\vec{x}_i - \vec{x}_j|} \tag{1}$$

where $|\vec{x}_i - \vec{x}_j|$ is the distance between the point $\vec{x}_i$ and $\vec{x}_j$. What is the potential energy of this system?

Let's start from a simple case. If $n = 2$, we have, from our earlier article,

$$U = k\frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|} = \frac{1}{2}\left(q_1\Phi(\vec{x}_1) + q_2\Phi(\vec{x}_2)\right) \tag{2}$$

If $n = 3$ we have three terms as follows:

$$U = k\left(\frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|} + \frac{q_1 q_3}{|\vec{x}_1 - \vec{x}_3|} + \frac{q_2 q_3}{|\vec{x}_2 - \vec{x}_3|}\right) = \frac{1}{2}\left(q_1\Phi(\vec{x}_1) + q_2\Phi(\vec{x}_2) + q_3\Phi(\vec{x}_3)\right) \tag{3}$$

If $n = 4$ we have six terms as follows:

$$\begin{aligned} U &= k\left(\frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|} + \frac{q_1 q_3}{|\vec{x}_1 - \vec{x}_3|} + \frac{q_1 q_4}{|\vec{x}_1 - \vec{x}_4|} + \frac{q_2 q_3}{|\vec{x}_2 - \vec{x}_3|} + \frac{q_2 q_4}{|\vec{x}_2 - \vec{x}_4|} + \frac{q_3 q_4}{|\vec{x}_3 - \vec{x}_4|}\right) & (4) \\ &= \frac{1}{2}\left(q_1\Phi(\vec{x}_1) + q_2\Phi(\vec{x}_2) + q_3\Phi(\vec{x}_3) + q_4\Phi(\vec{x}_4)\right) & (5) \end{aligned}$$

Now, we see the pattern. For general n, we have "n choose 2"(=$n(n-1)/2$) terms as follows:

$$U = k\sum_{i=1}^{n}\sum_{i<j}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \tag{6}$$

which is equal to

$$U = \frac{1}{2}\sum_{i=1}^{n} q_i\Phi(\vec{x}_i) \tag{7}$$

Let's turn this into a integral. If we define charge density ρ as follows,

$$q = \int dV\rho \tag{8}$$

where dV is the volume element, we have:

$$U = \frac{1}{2}\int \rho(\vec{x})\Phi(\vec{x})dV \tag{9}$$

Using the following Poisson's equation for electric potential,

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \tag{10}$$

(9) becomes:

$$U = -\frac{\epsilon_0}{2} \int \Phi \nabla^2 \Phi dV \tag{11}$$

Furthermore, integration by parts yields: (**Problem 1.** Check this! Hint[1])

$$U = \frac{\epsilon_0}{2} \int |\nabla \Phi|^2 dV = \frac{\epsilon_0}{2} \int |\vec{E}|^2 dV \tag{12}$$

In other words, u_E the energy density of electric field is given as follows:

$$u_E = \frac{\epsilon_0}{2} |\vec{E}|^2 \tag{13}$$

Problem 2. Consider a hollow sphere with radius r and charge q distributed homogeneously on the surface. What is its electric energy? Find it using two methods. First, use (9) and second, use (13). (Hint[2])

2 The energy density of magnetic field

Suppose a single coil in which an electric current I flows. Then, a magnetic field will be generated around the coil. If I changes the magnetic flux will change which in turn induces electromotive force (i.e. induced electric potential) in the circuit.

First, notice that if charge q rotates once in the circuit, the energy carried by the circuit will decrease as follows:

$$U = -q\mathcal{E} \tag{14}$$

Since the current is given as $I = dq/dt$ we have:

$$\frac{dU}{dt} = -I\mathcal{E} \tag{15}$$

We also know,

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d \oint \vec{A} \cdot d\vec{s}}{dt} \tag{16}$$

where $\vec{A}$ is the vector potential and $\vec{s}$ is the line element along the circuit. Plugging this into (15) yields:

$$\delta U = I \oint \delta \vec{A} \cdot d\vec{s} \tag{17}$$

where we changed d to δ. If the area of the cross section of coil is σ the current density is given by:

$$J = I/\sigma \tag{18}$$

[1] We have done a similar calculation in "A short introduction to quantum mechanics VII: the Hermiticity of the position operator and the momentum operator."

[2] For the second method use spherical coordinate.

Then, we have:

$$\delta U = J\sigma \oint \delta\vec{A}\cdot d\vec{s} \tag{19}$$

If we use the fact that the current density is parallel to $d\vec{s}$ as current is flowing along the circuit and the fact that area multiplied by the line element is the volume element, we can write:

$$\delta U = \int \delta\vec{A}\cdot\vec{J}dV \tag{20}$$

Given this, using Ampere's law, we can write:

$$\delta U = \frac{1}{\mu_0}\int \delta\vec{A}\cdot(\nabla\times\vec{B})dV \tag{21}$$

Using the following vector identity (see appendix of this article for a heuristic derivation),

$$\nabla\cdot(\vec{P}\times\vec{Q}) = \vec{Q}\cdot(\nabla\times\vec{P}) - \vec{P}\cdot(\nabla\times\vec{Q}) \tag{22}$$

we have:

$$\delta U = \frac{1}{\mu_0}\int[\vec{B}\cdot(\nabla\times\delta\vec{A}) + \nabla\cdot(\vec{B}\times\delta\vec{A})]dV \tag{23}$$

Upon using Stoke's theorem the second term vanishes if set $\vec{B}$ at infinity is zero. (This is similar in spirit to the case in electric field. If you solved Problem 1 correctly, you must have assumed $\vec{E} = 0$ at infinity.) Therefore, we get:

$$\delta U = \frac{1}{\mu_0}\int \vec{B}\cdot\delta\vec{B}dV \tag{24}$$

Using the following,

$$\delta(\vec{B}\cdot\vec{B}) = 2\vec{B}\cdot\delta\vec{B} \tag{25}$$

we obtain:

$$\delta U = \frac{1}{2\mu_0}\int \delta(\vec{B}\cdot\vec{B})dV \tag{26}$$

Therefore, we conclude:

$$U = \int\left(\frac{1}{2\mu_0}|\vec{B}|^2\right)dV \tag{27}$$

In other words, u_B the energy density of magnetic field is given as follows:

$$u_B = \frac{1}{2\mu_0}|\vec{B}|^2 \tag{28}$$

In conclusion, the total energy density of electromagnetic field is given as follows:

$$u = \frac{1}{2}\epsilon_0|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2 \tag{29}$$

Problem 3. Consider the electric field and the magnetic field that consist light as considered in "Light as electromagnetic waves." Show that in such a case, the energy density of electric field at a given point is equal to the energy density of magnetic field at the same point.

In our later article "A Relatively Short Introduction to General Relativity" we will derive this in another way from what is called "Lagrangian density" of electromagnetic field. Of course, there we have to assume how the Lagrangian density of electromagnetic field is given. Nevertheless, in our later article "non-Abelian gauge theory," we will see why the Lagrangian density of electromagnetic field must be given so. This fully justifies our later derivation.

Summary

- The energy density of electric field is E^2 multiplied by a certain factor, and the energy density of magnetic field is B^2 multiplied by another certain factor.

Appendix

∇ is a derivative operator. So, it respects Leibniz's rule. For example,

$$\nabla(fg) = (\nabla f)g + f(\nabla g) \tag{30}$$

$$\nabla \cdot (c\vec{h}) = (\nabla c) \cdot \vec{h} + c\nabla \cdot \vec{h} \tag{31}$$

The moral of the above equations is that when ∇ acts on a product of two objects, the answer is given by sum of two terms: the first term is given by product of the second object and the ∇ acting on the first object, and the second term is given by product of the first object and the ∇ acting on the second object.

In other words, if we write ∇_f meaning that ∇ acting only on f but not on g, ∇_g as ∇ acting only on g but not on f and so on, we can write the above equations redundantly as

$$\nabla(fg) = \nabla_f(fg) + \nabla_g(fg) = (\nabla_f f)g + f(\nabla_g g) \tag{32}$$

$$\nabla \cdot (c\vec{h}) = \nabla_c(c\vec{h}) + \nabla_h(c\vec{h}) = (\nabla_c c) \cdot \vec{h} + c\nabla_h \cdot \vec{h} \tag{33}$$

where we used $\nabla_f g = \nabla_g f = 0$ and so on because ∇_f acts only on f and not on g and vice versa.

Using this notation, we can write

$$\nabla \cdot (\vec{P} \times \vec{Q}) = \nabla_P \cdot (\vec{P} \times \vec{Q}) + \nabla_Q \cdot (\vec{P} \times \vec{Q}) \tag{34}$$

Using $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B})$, the first term can be re-written as

$$\nabla_P \cdot (\vec{P} \times \vec{Q}) = \vec{Q} \cdot (\nabla_P \times \vec{P}) = \vec{Q} \cdot (\nabla \times \vec{P}) \tag{35}$$

Similarly, the second term can be re-written as (**Problem 4.** Show this! Hint[3])

$$\nabla_Q \cdot (\vec{P} \times \vec{Q}) = -\vec{P} \cdot (\nabla_Q \times \vec{Q}) = -\vec{P} \cdot (\nabla \times \vec{Q}) \tag{36}$$

Summing (35) and (36), we get (22).

Problem 5. Obtain (22), using a rigorous method. (Hint[4])

[3] See "The cross product revisited."

[4] Use $\nabla \cdot (\vec{P} \times \vec{Q}) = \epsilon_{ijk}\partial^i(P^j Q^k)$.