

Entropy changes, an example

Earlier, we have explained that entropy always increases and provided a formula to calculate the entropy change. Now, let's actually calculate it in some examples.

Suppose you have two objects with $m = 1$ kg each. They have the same specific heat $c = 200$ J/kg·K. If one is 299.7 K, and another is 300.3 K. What is the total entropy increase as they reach thermal equilibrium?

The thermal equilibrium is reached at 300 K. The initial temperature of the first object is 299.7 K and the final temperature is 300 K. Thus,

$$\begin{aligned}\Delta S_1 &= \int \frac{dQ}{T} = \int_{299.7}^{300} \frac{mcdT}{T} = mc \ln \left(\frac{300}{299.7} \right) \\ &= 200 \ln \left(\frac{1}{0.99} \right) \text{ J/K} = -200 \ln 0.99 \text{ J/K}\end{aligned}\quad (1)$$

which is positive, as it gained energy. The initial temperature of the second object is 300.3 K and the final temperature is 300 K. Thus,

$$\Delta S_2 = \int \frac{dQ}{T} = \int_{300.3}^{300} \frac{mcdT}{T} = mc \ln \left(\frac{300}{300.3} \right) = -200 \ln 1.01 \text{ J/K} \quad (2)$$

which is negative, as it lost energy.

Therefore, the total entropy change is

$$\Delta S = \Delta S_1 + \Delta S_2 = -200 \ln 0.9999 \text{ J/K} \approx 0.02 \text{ J/K} \quad (3)$$

Now, let me explain how *not* to solve this problem. We know that $\Delta S = \Delta Q/T$. As the difference between 299.7 K and 300 K is not big, if you ignore the fact that T changes during the thermal exchange, and just plug in $T = 299.7$ K which is the initial temperature, you get

$$\Delta S_1 = \int \frac{dQ}{T} = \frac{mc\Delta T}{299.7} = \frac{60}{299.7} \text{ J/K} \quad (4)$$

denoting $\Delta T = 0.3$ K. For the second object, if you ignore the difference between 300.3 K and 300 K, and just plug in 300.3 K, the initial temperature, for T , you get

$$\Delta S_2 = \int \frac{dQ}{T} = \frac{mc\Delta T}{300.3} = -\frac{60}{300.3} \text{ J/K} \quad (5)$$

$$\Delta S = \Delta S_1 + \Delta S_2 \approx 0.04 \text{ J/K} \quad (6)$$

So, you get a wrong answer. You see that you can never ignore the variation of T in the denominator of $\frac{dQ}{T}$ and set T for its initial value. Then, would it be better if you set the denominator of $\frac{dQ}{T}$ for its final value 300 K? Then, you will see that

$$\Delta S_1 = 60/300 \text{ J/K} = 0.2 \text{ J/K}, \quad \Delta S_2 = -60/300 \text{ J/K} = -0.2 \text{ J/K} \quad (7)$$

So, the total entropy change would be zero! It is worse than the first approximation.

Let's look at this all again slightly more closely. Let's say $T = 300 \text{ K}$ and $\Delta T = 0.3 \text{ K}$. Then, (1), (2) and (3) are

$$\Delta S_1 = mc \ln \left(\frac{T}{T - \Delta T} \right), \quad \Delta S_2 = mc \ln \left(\frac{T}{T + \Delta T} \right) \quad (8)$$

$$\Delta S = \Delta S_1 + \Delta S_2 = mc \ln \left(\frac{T^2}{T^2 - \Delta T^2} \right) \quad (9)$$

Notice that (8) can be approximated, to first order in ΔT , as

$$\Delta S_1 \approx mc \frac{\Delta T}{T}, \quad \Delta S_2 \approx -mc \frac{\Delta T}{T} \quad (10)$$

which is exactly our second approximation (7). (**Problem 1.** Show (10) again by using $\ln(1+x) \approx x$.)

As the total entropy change, $\Delta S_1 + \Delta S_2 = 0$, in the first order in ΔT , we see that it should be at least in the second order or higher. Actually, you can easily check that (9) is the second order in ΔT .

Problem 2. Obtain the first term in Taylor series for (9) (i.e., the term which is second order in ΔT).

Summary

- Use

$$dS = \frac{dQ}{T}$$

to calculate the entropy change.