What is enumerative geometry?

I have read "Enumerative Geometry and String Theory" by Sheldon Katz and found it fascinating. Enumerative geometry is a branch of algebraic geometry that is also in a close relation with string theory. Understanding string theory requires an advanced knowledge of physics, but enumerative geometry itself doesn't, so I will explain what enumerative geometry is in this article; I want to note that understanding what enumerative geometry is about is much easier than understanding enumerative geometry itself.

We should start by understanding conic sections. As mentioned in the last article, any conic section can be described by a 2nd polynomial as follows:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \text{ with } A, B, C \text{ not all zero} \tag{1}$$

For example,

- If $A = 1, B = 0, C = 0, D = 0, E = -1, F = 0$, we get $y = x^2$, which is a parabola.
- If $A = 1, B = 0, C = 1, D = 0, E = 0, F = -1$, we get $x^2 + y^2 = 1$, which is a circle.
- If $A = 2, B = 0, C = 1, D = 0, E = 0, F = -1$, we get $2x^2 + y^2 = 1$, which is an ellipse.
- If $A = 0, B = 1, C = 0, D = 0, E = 0, F = -1$, we get $xy = 1$, which is a hyperbola.

We may now ask the following question: If there are m points in a plane, how many conic sections pass through all of these points? To answer this problem, let's denote the coordinates of these m points by $(x_1, y_1), (x_2, y_2), \cdots (x_m, y_m)$, and plug these coordinates into the degree 2 polynomial defined above. For m points, we will have m equations:

$$Ax_i^2 + Bx_iy_i + Cy_i^2 + Dx_i + Ey_i + F = 0 \tag{2}$$

where i runs from 1 to m.

So we have m equations to solve, while there are six variables. However, our six variables are really only five, because

$$kAx^2 + kBxy + kCy^2 + kDx + kEy + kF = 0 \tag{3}$$

describes the same curve as $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. In other words, if A is not zero, we can write the above equation as

$$x^2 + bxy + cy^2 + dx + ey + f = 0 \tag{4}$$

where $b = B/A$, $c = C/A$, $d = D/A$, $e = E/A$, $f = F/A$. Therefore, we see that there are only five variables (b, c, d, e, f) instead of the redundant six. If A is zero, then we can divide the equation (1) by B instead of A, and get five variables. If B is also zero, we can divide the equation (1) by C, and this can be always done since by definition A, B, and C are not all zero.

As conic sections are described by five variables (or six, one of which is redundant), if we have five equations about them, we will have one solution in the generic case. If there are more than five equations, then there will be no answer in generic case. If there are less than five equations, then there will be infinitely many answers in generic case.

Therefore, we conclude that there are infinitely many conic sections that pass through less than five general points; there is no conic section that passes through more than five general points; and there is one conic section which passes through five general points.

Now, let's extend our analysis to the case where a conic section passes through m points and is tangent to n lines. As this gives $(m+n)$ constraints (i.e., $(m + n)$ equations), there are infinitely many such conic sections in generic cases where $m + n$ is less than five, while there is no such conic section in generic cases, where $m + n$ is more than five.

So, the critical case is when $m + n$ is five. Given this, we may guess that there is only one conic section which passes through m general points and is tangent to general $(5 - m)$ lines. However, this is too naive, and is not the case. In fact, in general cases, when $m = 1$, and $n = 4$, we have two such curves, and when $m = 2$, $n = 3$, we have four such curves, and when $m = 3$, $n = 2$, we have four such curves, and when $m = 4$, $n = 1$ we have two such curves, and finally when $m = 5$, $n = 0$ we have one such curve.

How this can be calculated is beyond the scope of this article. Instead, I want to give a heuristic reason why five equations (m equations for points and $(5 - m)$ equations for lines) for five variables don't translate into a single solution. This may not be intuitive, but it is not a new concept if you consider that a quadratic equation for one variable can yield two (not one) solutions. Likewise, a quadratic equation can be factored into two linear equations, and if an equation satisfies either one of them, then it is a solution, which means that there can be two solutions. Extending this reasoning, there can be more than one solutions to these five equations.

Another interesting fact you might have noticed is that the number of conic sections that pass through m given points and is tangent to n given lines is the same as the number of conic sections that pass through n given points and is tangent to m given lines. This is not a coincidence; it is actually due to duality, on which I will not elaborate here.

A final remark: The numbers of conic sections that pass through m general points and are tangent to $(5-m)$ general lines are called characteristic numbers. And "the problem of computing characteristic numbers was formulated in the 19th century, but it was found for plane curves of arbitrary degree only after the techniques of stable maps inspired by physics were developed" (p 40, "Enumerative Geometry and String Theory"). Remember that we only considered finding the characteristic numbers for plane curves of degree 2 in this article; the above quote implies that the characteristic numbers for plane curves of degree 3 or higher was solved only after the techniques inspired by string theory was developed. It is remarkable that string theory motivates and influences mathematics to such a great degree. The string theorist Edward Witten, who won the Fields Medal (the Nobel prize equivalent in mathematics) once claimed that the mathematics of the third millennium will be dominated by string theory.

Summary

- Enumerative geometry is a branch of algebraic geometry that is also in a close relation with string theory.