Equivalence relation

Equivalence relation is often used in mathematics, and sometimes in physics. While it is not usually covered in high school math or any other low-level math courses, I explain it here because it is interesting enough and easy enough, and useful.

"$\sim$" is called an equivalence relation if it satisfies all of the three following conditions.

- $a \sim a$.
- if $a \sim b$ then $b \sim a$.
- if $a \sim b$ and $b \sim c$ then $a \sim c$.

If $a \sim b$ we say a and b belong to the same "equivalence class." If $a \sim b$ is not satisfied, a and b belong to two different equivalence classes. An arbitrary object a belongs to only one equivalence class. It cannot belong to two or more equivalence classes at the same time. We usually denote an equivalence class by one of its member. For example, the equivalence class a belongs to is often represented by $[a]$.

The easiest example of an equivalence relation is the equal sign "$=$". It is easy to check the three conditions. $a = a$ is always satisfied. If $a = b$ then $b = a$. If $a = b$ and $b = c$ then $a = c$.

Another example of an equivalence relation is following.

$$a \sim b, \quad \text{if } a - b = 3n \text{ for some integer } n \tag{1}$$

For example, we have $4 \sim 10$ because $4 = 10 + 3(-2)$. Let's check that the three conditions are satisfied. The first condition is satisfied

$$a = a + 3(0) \tag{2}$$

as 0 is an integer. The second condition is also satisfied from the following reason.

$$\text{if } a - b = 3n \ \text{ then } b - a = 3(-n) \tag{3}$$

as $-n$ is an integer if n is also integer. The third condition is also satisfied from the following reason.

$$\text{if } a - b = 3n, \quad b - c = 3m \ \text{ then } a - c = 3(n + m) \tag{4}$$

as $n+m$ is also an integer, if n and m are integers.

Thus, (1) is indeed a good equivalence relation. In other words, if you divide two numbers by 3 and their remainders are the same, they belong to the same equivalence class. In other words, the remainder determines the equivalence class.

Problem 1. How many equivalence classes are there for the whole set of integers, if equivalence classes are defined by (1)?

Mathematicians often denote the equivalence class just introduced by the following notation:

$$a \equiv b \pmod 3 \tag{5}$$

For example,

$$7 \equiv 1 \pmod 3 \tag{6}$$

$$5+2 \equiv 1 \pmod 3 \tag{7}$$

We pronounce mod 3 as "modulo three." Actually, there is no reason why only 3 needs to be modulus. Any positive integer can be modulus. For example, we have

$$15 \equiv 7 \pmod 4 \tag{8}$$

because $15-7$ is divisible by 4. We easily see that mod n for any positive integer n is an equivalence relation.

We have actually encountered some other examples of equivalence relations in our earlier articles on triangles even though we didn't explictly use the word "equivalence relation." Congruency of triangle and similarity of triangle are equivalence relations. That the congruency of triangle is an equivalence relation can be checked as follows.

> Triangle A is congruent to triangle A. If triangle A is congruent to triangle B then triangle B is congruent to triangle A. If triangle A is congruent to triangle B and triangle B is congruent to triangle C then triangle A is congruent to triangle C.

The above three sentences are all true statements. That the similarity of triangle is an equivalence relation can be checked similarly.

An example that is *not* an equivalence relation is $\approx$ (approximately equal to). At first glance, it seems to be an equivalence relation. As the first and the second conditions are trivially satisifed, let's just check the third condition with an example. As we have $\sqrt{9.81} \approx 3.14$ and $3.14 \approx \pi$ we indeed have $\sqrt{9.81} \approx \pi$. It seems to work. However, not always. Let's repetitively apply the third condition.

$$3 \approx 3.0001 \approx 3.0002 \approx 3.0003 \cdots 99.9999 \approx 100 \approx 100.0001 \tag{9}$$

If the third condition were true we would have $3 \approx 100.0001$ which is not true! Therefore, we have shown that $\approx$ is not an equivalence relation. This conclusion cannot be avoided even if we precisely define $\approx$ say, “agrees within 10%.”

Problem 2. Answer whether each of the three following relations is an equivalent relation. If it is not, give a counter-example (i.e., an example that shows that it is not.)

- $a \sim b$ if $\sin a = \sin b$.
- $a \sim b$ if $a = b + 3n + 1$ for some integer n.
- $a \sim b$ if $a \geq b$.

Summary

- An equivalence relation satisfies the following three conditions.
- 1) $a \sim a$.
- 2) if $a \sim b$ then $b \sim a$.
- 3) if $a \sim b$ and $b \sim c$ then $a \sim c$.