Even functions and odd functions

Even functions are any functions that satisfy the following criteria:

$$f(-x) = f(x) \tag{1}$$

Some examples of even functions are $2x^2$, $3x^4$, $4x^6 - x^8$, $\cos x$. The graph of an even function is symmetric with respect to y-axis. See Fig.1 for a graph of an even function.

Odd functions are any functions that satisfy the following criteria:

$$f(-x) = -f(x) \tag{2}$$

Some examples of odd functions are $2x^3$, $3x$, $4x^3 - x^5$, $\sin x$. The graph of an odd function remains unchanged when rotated around the origin by 180 degrees. See Fig.2 for a graph of an odd function.

If you Taylor-expand an even function around $x = 0$, only terms with even power are present. This is the reason why even functions are called even functions. It is easy to check that the presence of odd powers make the function not satisfy (1). For example, if we have $f(x) = x^2 + 2x^3 + x^4 + x^6 \cdots$, then:

$$f(-x) = x^2 - 2x^3 + x^4 + x^6 \cdots \neq f(x) = x^2 + 2x^3 + x^4 + x^6 \cdots \tag{3}$$

Similarly, if you Taylor-expand an even function around $x = 0$, only terms with odd power are present. This is the reason why odd functions are called odd functions. It is easy to check that the presence of even powers make the function not satisfy (2). For example, if we have $f(x) = x + 2x^2 + x^3 - x^5 \cdots$, then:

$$f(-x) = -x + 2x^2 - x^3 + x^5 \cdots \neq -f(x) = -x - 2x^2 - x^3 + x^5 \cdots \tag{4}$$

Also, odd functions always satisfy $f(0) = 0$ since plugging $x = 0$ to (2) yields $f(0) = -f(0)$.

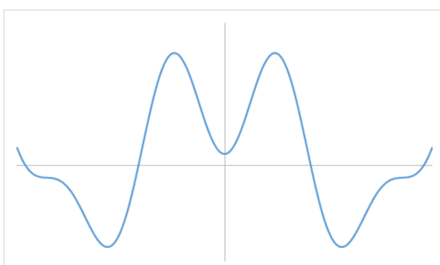

Figure 1: an even function

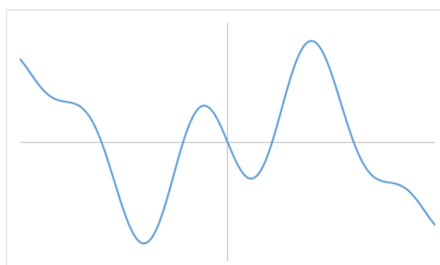

Figure 2: an odd function

Problem 1. Which ones of the following are even functions? odd functions? neither? both?

$$\sin x,\ \ \cos x,\ \ \sin x\cos x,\ \ \sin x^2,\ \ \cos x^2,\ \ \sin x+\cos x$$

$$\sin(\sin x),\ \ \sin(\cos x),\ \ \cos(\cos x),\ \ \cos(\sin x),\ \ \tan x,\ \ \sin^2 x$$

$$\cosh x,\ \ \sinh x,\ \ 0,\ \ 3,\ \ x\sin x,\ \ x\cos x,\ \ x+\cos x,\ \ x+\sin x$$

Problem 2. Prove that an arbitrary function $f(x)$ can be expressed as a sum of an even function $g(x)$ and an odd function $h(x)$. Prove this by explicitly obtaining $g(x)$ and $h(x)$ in terms of $f(x)$.

Problem 3. Prove the following, given that $f(x)$ is an odd function.

$$\int_{-a}^{a} f(x)dx = 0 \tag{5}$$

Summary

- Even functions satisfy $f(-x) = f(x)$.
- Odd functions satisfy $f(-x) = -f(x)$.
- Any function is expressible as a sum of an even function and an odd function.