Even permutation, odd permutation, and $\det A^T = \det A$

As we introduced in our earlier article, the symmetric group S_n is a group of all permutations of a set of n elements. In our earlier articles "Bosons, Fermions, and Pauli's exclusion principle" and "Levi-Civita Symbol," we also learned the concepts of "even permutation" and "odd permutation." In this article, we will deal with even permutation and odd permutation rigorously. We will also prove $\det A^T = \det A$.

Problem 1. Let's consider S_5. From an earlier article, we know that

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & b & c & d & e \end{pmatrix} \tag{1}$$

is an even permutation if $\epsilon_{abcde} = 1$ and an odd permutation if $\epsilon_{abcde} = -1$. It is common to denote an even permutation by $\mathrm{sgn}(\sigma) = 1$ and an odd permutation by $\mathrm{sgn}(\sigma) = -1$. Now, let's define a mapping D by $D(\sigma) = \mathrm{sgn}(\sigma)$. Show that D is a (1×1 matrix) representation of S_5.

Now, we would like to show that the determinant of the transpose of a matrix is equal to the determinant of the original matrix. In other words, $\det(A^T) = \det A$. We will consider a 4×4 matrix as an example. First, recall

$$\det A = \sum_{i,j,k,l} \epsilon_{ijkl} A_{i1} A_{j2} A_{k3} A_{l4} \tag{2}$$

which implies

$$\det A^T = \sum_{i,j,k,l} \epsilon_{ijkl} A_{1i} A_{2j} A_{3k} A_{4l} \tag{3}$$

Here, you can easily see that each contribution to the sum is zero if ϵ_{ijkl} is zero. In other words, i,j,k,and l have to be some permutations of 1, 2, 3, and 4, for to give non-zero contributions to the sum. In other words, there are 4! terms in (2), and also 4! terms in (3). If each term in (2) and (3) can be matched together and is the same, the sums are also the same.

For this purpose, let's re-express (2) as

$$\det A = \sum_{\text{all } \sigma} \mathrm{sgn}(\sigma) A_{\sigma(1)1} A_{\sigma(2)2} A_{\sigma(3)3} A_{\sigma(4)4} \tag{4}$$

and (3) as

$$\det A^T = \sum_{\text{all } \sigma'} \text{sgn}(\sigma') A_{1\sigma'(1)} A_{2\sigma'(2)} A_{3\sigma'(3)} A_{\sigma'(4)4} \tag{5}$$

Given this, we want to match the term that is

$$A_{\sigma(1)1} A_{\sigma(2)2} A_{\sigma(3)3} A_{\sigma(4)4} = A_{1\sigma'(1)} A_{2\sigma'(2)} A_{3\sigma'(3)} A_{4\sigma'(4)} \tag{6}$$

In other words,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) \end{pmatrix} = \begin{pmatrix} \sigma'(1) & \sigma'(2) & \sigma'(3) & \sigma'(4) \\ 1 & 2 & 3 & 4 \end{pmatrix} \tag{7}$$

Problem 2. Show that $\sigma = \sigma'^{-1}$. Thus, we can say $\sigma^{-1} = \sigma'$.

Problem 3. Show that $\text{sgn}(\sigma^{-1}) = \text{sgn}(\sigma)$. (Hint[1]) Thus, we can say $\text{sgn}(\sigma') = \text{sgn}(\sigma)$

Therefore, (6) is promoted to

$$\text{sgn}(\sigma) A_{\sigma(1)1} A_{\sigma(2)2} A_{\sigma(3)3} A_{\sigma(4)4} = \text{sgn}(\sigma') A_{1\sigma'(1)} A_{2\sigma'(2)} A_{3\sigma'(3)} A_{4\sigma'(4)} \tag{8}$$

As all the terms in (2) and (3) are the same, we conclude $\det A^T = \det A$.

Summary

- $\det(A^T) = \det A$

[1] Show first $\text{sgn}(\sigma)\text{sgn}(\sigma^{-1}) = 1$ by using the fact that sgn is a representation.