## Exponential functions and natural logs

Let's calculate the derivative of $y=a^{x}$. We have:

$$
\begin{equation*}
\frac{d y}{d x}=\lim _{\Delta x \rightarrow 0} \frac{a^{x+\Delta x}-a^{x}}{\Delta x}=a^{x}\left(\lim _{\Delta x \rightarrow 0} \frac{a^{\Delta x}-1}{\Delta x}\right) \tag{1}
\end{equation*}
$$

Therefore, if we can calculate the term in the parenthesis, we can calculate the derivative. Notice also that if this term is 1 , the derivative of $a^{x}$ for such an $a$ is equal to the original function. This fact plays a very important role in mathematics. Therefore, mathematicians denote such $a$ with a special symbol " $e$ ". In other words,

$$
\begin{equation*}
\lim _{\Delta x \rightarrow 0} \frac{e^{\Delta x}-1}{\Delta x}=1 \tag{2}
\end{equation*}
$$

However, it would be hard to calculate the value of $e$ using this definition. We will later find another definition which is easier to calculate. In any case, we have

$$
\begin{equation*}
\text { if } y=e^{x}, \quad \frac{d y}{d x}=e^{x}=y \tag{3}
\end{equation*}
$$

$e^{x}$ is called "exponential function." Sometimes it is expressed as $\exp x$. Also, because of the special status of $e$, it turns out that it is very useful to use logarithm to the base $e$. We call it natural logarithm and denote it as $\ln$ because natural logarithm is "logarithme naturel" in French. However, many mathematicians and physicists use log to denote ln as well.

Now, let's calculate the derivative of natural log. From (3), we have:

$$
\begin{equation*}
x=\ln y, \quad \frac{d x}{d y}=\frac{1}{y} \tag{4}
\end{equation*}
$$

So, we find that the derivative of natural $\log$ is very simple. Now, let's try to derive this formula by another way. We have:

$$
\begin{align*}
\frac{d x}{d y} & =\lim _{\Delta y \rightarrow 0} \frac{\ln (y+\Delta y)-\ln y}{\Delta y}=\lim _{\Delta y \rightarrow 0} \frac{\ln \left(1+\frac{\Delta y}{y}\right)}{\Delta y}  \tag{5}\\
& =\frac{1}{y} \lim _{\Delta y \rightarrow 0} \frac{y}{\Delta y} \ln \left(1+\frac{\Delta y}{y}\right)  \tag{6}\\
& =\frac{1}{y} \lim _{\Delta y \rightarrow 0} \ln \left(\left(1+\frac{\Delta y}{y}\right)^{y / \Delta y}\right)  \tag{7}\\
& =\frac{1}{y} \lim _{\Delta y / y \rightarrow 0} \ln \left(\left(1+\frac{\Delta y}{y}\right)^{1 /(\Delta y / y)}\right)  \tag{8}\\
& =\frac{1}{y} \ln \left(\lim _{t \rightarrow 0}(1+t)^{1 / t}\right) \tag{9}
\end{align*}
$$

Comparing with (4), we conclude:

$$
\begin{array}{r}
1=\ln \left(\lim _{t \rightarrow 0}(1+t)^{1 / t}\right) \\
e=\lim _{t \rightarrow 0}(1+t)^{1 / t} \tag{11}
\end{array}
$$

Therefore, we have obtained another definition of $e$. Notice that the limit in the above formula is neither 1 nor infinity. In the limit, as $1+t$ goes to 1 , at first glance, it may seem that the value obtained could be 1 since 1 to the power of any number is 1 . However, we are multiplying it in an infinite number, so it's actually bigger than 1 . On the other hand, the power $(1 / t)$ goes to infinity so it may seem that the result is infinity since if you multiply any number bigger than 1 infinite times, you get an infinity. However, again, we are multiplying a number barely bigger than 1 , so it doesn't become infinity. Anyhow, $e$ is $2.71828 \cdots$.

Final remark. From (4), we have:

$$
\begin{equation*}
\frac{d(\ln y)}{d y}=\frac{1}{y} \tag{12}
\end{equation*}
$$

In other words,

$$
\begin{equation*}
\frac{d(\ln x)}{d x}=\frac{1}{x} \tag{13}
\end{equation*}
$$

We indeed see that $e$ is an important number.
Problem 1. What is the domain of the function $e^{x}$ ? What is its range? (Hint ${ }^{1}$ )
Problem 2. What is the domain of the function $\ln x$ ? What is its range? (Hint ${ }^{2}$ )
Problem 3. Prove

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left(1+\frac{1}{t}\right)^{t}=e \tag{14}
\end{equation*}
$$

Problem 4. Prove

$$
\begin{equation*}
\lim _{t \rightarrow 0}(1+t)^{a / t}=e^{a} \tag{15}
\end{equation*}
$$

Using this relation, prove "law of seventy" which was explained in our earlier article "Logarithm." (Hint ${ }^{3}$ )

Problem 5. $\left(\right.$ Hint $\left.^{4}\right)$

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left(1+\frac{1}{t}\right)^{t^{2}}=?, \quad \lim _{t \rightarrow \infty}\left(1+\frac{1}{t^{2}}\right)^{t}=? \tag{16}
\end{equation*}
$$

Problem 6.

$$
\begin{gather*}
\left(e^{x^{2}+2 x}\right)^{\prime}=?, \quad\left(e^{-x}\right)^{\prime}=?  \tag{17}\\
\left(x e^{-x}\right)^{\prime}=? \tag{18}
\end{gather*}
$$

[^0]Problem 7. Prove the following. (Hint ${ }^{5}$ )

$$
\begin{equation*}
\left(a^{x}\right)^{\prime}=\ln a \cdot a^{x} \tag{19}
\end{equation*}
$$

Problem 8. Prove the following applying l'Hôpital's rule 10 times.

$$
\begin{equation*}
\lim _{x \rightarrow \infty} \frac{1.2^{x}}{x^{10}}=\infty \tag{20}
\end{equation*}
$$

Similarly, one can easily show that an exponential function always grows faster than any polynomial function.

Problem 9. $\left(\right.$ Hint $^{6}$ )

$$
\begin{equation*}
\lim _{x \rightarrow 0^{+}} x \ln x=?, \quad \lim _{x \rightarrow 0^{+}} x^{x}=? \tag{21}
\end{equation*}
$$

Problem 10. Find the value of $x$ when $x^{x}$ is minimum. (Hint ${ }^{7}$ )
Problem 11. Let $f(x)=x^{5}+2 x^{3}+x+2$, and let $g(x)=f^{-1}(x)$. Then, we have $f(1)=5$ and $g(5)=1$. By calculating $f^{\prime}(1)$ obtain $g^{\prime}(5) .\left(\right.$ Hint $\left.^{8}\right)$

## Summary

- $\frac{d e^{x}}{d x}=e^{x}$
- $\frac{d(\ln x)}{d x}=\frac{1}{x}$

[^1]
[^0]:    ${ }^{1}$ If you don't know the term "domain" and "range," read our earlier article "What is a function?" Also, think whether $e^{x}$ can be negative or zero.
    ${ }^{2}$ The answer to problem 1 is helpful.
    ${ }^{3}$ Set $a=\ln 2$
    ${ }^{4}$ For the first one, use (14). For the second one, use $\left(1+1 / t^{2}\right)^{t}=\left(1+1 / t^{2}\right)^{t^{2} / t}$.

[^1]:    ${ }^{5}$ Use $a^{x}=e^{(\ln a) x}$.
    ${ }^{6}$ Use $x \ln x=\frac{\ln x}{1 / x}$ and l'Hôpital's rule. For the second one use $e^{x \ln x}=x^{x}$.
    ${ }^{7}$ Use $x^{x}=e^{x \ln x} . x^{x}$ is minimum when $x \ln x$ is
    ${ }^{8}$ Remember how we calculated the derivative of natural log in (4).

