Exponential function vs Polynomial function

Which function grows faster, 1.2^x or x^{10}? To answer this question, let's plot these function. See Fig.1. 1.2^x is plotted as thick line and x^{10} is plotted as thin line. We see that x^{10} is skyrocketing while 1.2^x is increasing gradually. So, the case seems to be closed. x^{10} seems to grow faster.

However, this is not the full picture. What will happen when x is bigger? In the figure, the functions are only plotted up to $x = 6$. Something different may happen when x is bigger.

Indeed, it turns out that 1.2^x catches up x^{10} when x is $315.627\cdots$ and eventually grows much bigger than x^{10}. Why is this so? Let's analyze this quantitatively.

First, let's see how long it takes for each function to double. To this end, assume that the functions start out from $x = x_0$ and doubled when $x = x_0 + \Delta x$. In other words, it takes Δx to be doubled if we start from x_0.

Let's obtain explicit value for Δx. In case of 1.2^x, we have:

$$1.2^{x_0+\Delta x} = 2 \times 1.2^{x_0} \tag{1}$$

Problem 1. Show

$$\Delta x = \log_{1.2} 2 = \frac{\ln 2}{\ln 1.2} \tag{2}$$

If you numerically calculate this, it is about 3.80.

In case of x^{10}, we have:

$$(x_0 + \Delta x)^{10} = 2 \times x^{10} \tag{3}$$

Problem 2. Show

$$\Delta x = (2^{1/10} - 1)x_0 \tag{4}$$

If you numerically calculate this, it is about $0.07x_0$.

Now, let's analyze. When x_0 is 1, 1.2^x takes about 3.80 to double while x^{10} takes about 0.07 to double. Indeed, 1.2^x grows faster for such a small x_0. However, when x_0 is 100, 1.2^x still takes about 3.80 to double while x^{10} takes about 7 to double. It shows that 1.2^x grows more rapidly for this value of x_0. Notice that 1.2^x always takes only 3.80 to double, while a very big Δx is needed to double x^{10} for a very big x_0. In other words, the bigger x_0, the longer it takes to double x^{10}. It implies that it is destined that 1.2^x eventually takes up x^{10} and becomes infinitely bigger than x^{10} for an infinitely big x. Therefore, in such a case, mathematicians say "1.2^x grows faster than x^{10}." When they say that it grows faster it is implicitly implied that the case is for big x.

In a broad sense, function of the form $f(x) = c \cdot a^x$ is called an exponential function as the variable x is the exponent. In a narrow sense, function of the form $f(x) = e^x$ is called an

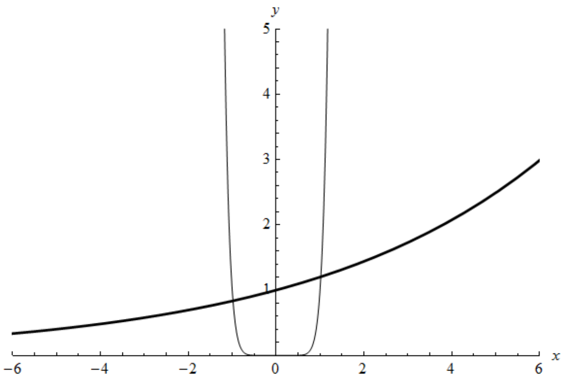

Figure 1: 1.2^x and x^{10}

exponential function, where e is the number introduced in our earlier article "Logarithm." And, you already know what a polynomial function is from our earlier article "Polynomials, expansion and factoring." Generalizing our arguments in this article, one can show that an exponential function always grows faster than a polynomial function, as long as c is positive and a is bigger than 1. For example, $0.00001 \cdot 1.00001^x$ grows faster than $x^{10000000}$. In our later article "Exponential function and natural log," you will be invited to prove in an alternative way that an exponential function always grows faster than a polynomial function. In any case, in our later article "Approximation of the naive black hole degeneracy," we will see how the fact that an exponential function always grows faster than a polynomial function is related to black hole entropy.

Problem 3. Which of the following two functions approach to 0 faster? e^{-10x} or $\dfrac{1}{x^{10}}$?

Summary

- An exponential function grows faster than a polynomial function.