Velocity revisited and extremum

Suppose you throw a ball upward and its hight hm at time $t$sec is given by the following formula:

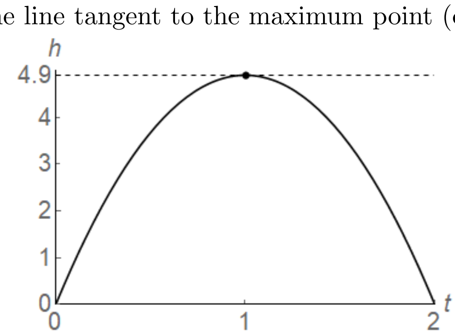
$$h = 9.8t - 4.9t^2 \tag{1}$$

Given this, let me ask you a question. When does it reach the highest point? If you know some properties of quadratic functions, you may easily answer this question. But, in this article, we will look at it differently, by using calculus, even though the result is same.

To this end, first notice that the velocity of the ball is positive when it is moving upward and negative when it is moving downward. Therefore, it is easy to see that the velocity of ball must be momentarily zero when it is changing its direction. This is also when it is at the highest point. Therefore, we obtain:

$$\frac{dh}{dt} = 9.8 - 4.9 \cdot 2t = 9.8 - 9.8t = 0 \tag{2}$$

We conclude that the ball reaches the highest point when $t = 1$. See the figure below. We see that the derivative of the function $h(t)$ is zero, at its maximum point. You can see that the line tangent to the maximum point (denoted as the dotted line) has slope zero.



Now, let's move onto a similar problem. Consider the following function:

$$f(x) = x^4 - 2x^2, \qquad (-1.5 \leq x \leq 1.5) \tag{3}$$

See, also Fig. 1. It looks like a Mexican hat. Therefore, physicists call a potential (i.e. potential energy) that looks like this function "Mexican hat potential" or "Higgs potential." Yes, the same Higgs who got a Nobel Prize in physics for the prediction of the presence of Higgs boson (also called "Higgs particle.") Higgs potential is a sum of two terms: the quartic term (i.e. the one proportional to x^4) and the quadratic term (i.e. the one proportional to x^2). The coefficient for the quartic term is positive (in this case 1) while the coefficient for the quadratic term is negative (in this case -2.) Otherwise, the function never looks like a Mexican hat.

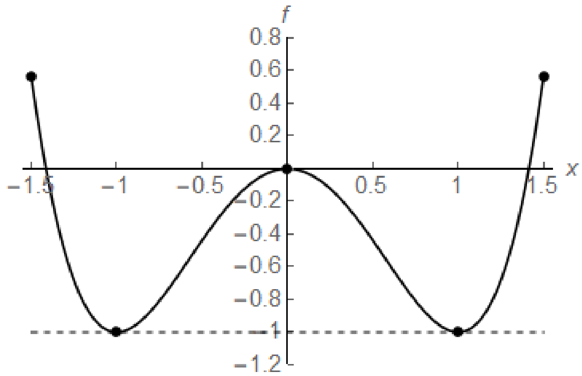


Figure 1: $f(x)$

When is this function maximum? From Fig. 1 it is obvious that this is achieved when $x = 1.5$ or $x = -1.5$. When is it minimum? From the figure, we can see that this is achieved when x is around -1 or 1. Here, notice that the derivatives of $f(x)$ are zero, when $f(x)$ is at minimum; in the figure, you see the dotted line which is tangent to the graph at the two points $(-1, -1)$ and $(1, -1)$ has a zero slope. Think it along this way. If you start from $x = -1.5$ and increase x bit by bit, f will decrease bit by bit until $f' = 0$ (i.e. $x = -1$), at which f is minimum. Then, the f will start to increase again. Thus, you can understand that the slope of the function f must be zero at its minimum. Similar can be said about the point $x = 1$.

Remember that in our first example in this article, we have seen that the maximum of the function (1) is achieved when its derivative is zero. Here, we see that the minimum of the function (3) is achieved when its derivative is zero. If you want to find the maximum and the minimum of a function, it seems like it's a good strategy to look for the points where its derivative is zero.

Actually, if you want to find the maximum or minimum of function f defined over $a \leq x \leq b$, you have to find the value of the function when its derivative vanishes, and the value at its endpoints (i.e. $f(a)$, $f(b)$). The largest of these values is the maximum and the smallest one the minimum.

Let's apply this to our example here. Let's first find the points where the derivative is zero. We obtain:

$$f'(x) = 4x^3 - 4x = 0 \tag{4}$$

$$x = -1, 0, 1 \tag{5}$$

However, from the figure it is clear that the function is neither minimum nor maximum when x is zero, even though its derivative is zero. And, we have $f(-1) = f(1) = -1$. These are the minimums of the function f.

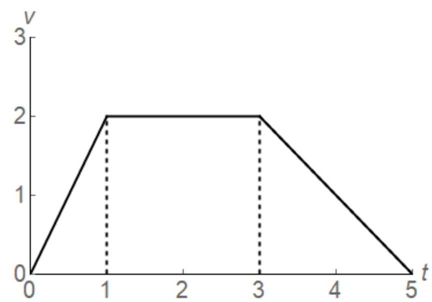
Again, $x = 0$ is not maximum even though $f' = 0$ as $f(1.5)$ is bigger. However, we call

$f(x = 0)$ a "local maximum" since it is the maximum around its small neighborhood; say, when $-0.1 \le x \le 0.1$, the point $x = 0$ is clearly the maximum. Similarly, $x = -1$ and $x = 1$ are also local minimums, while they are the (global) minimums as well. (A "global" minimum is the true minimum in the whole specified range.) We call minimum or maximum "extremum." It is also easy to see that the derivative of a function must be zero, for it to be a local extremum.

Summarizing, we can re-cast our earlier statements as follows: Only the values at the endpoints, and the local extremums can be the (global) extremum.

Problem 1. Suppose a car is moving along a straight line, and at t seconds, the position of the car is given by $s(t) = t^3 - 9t^2 + 24t + 5$. What is the velocity of the car in terms of t? What is the acceleration of the car in terms of t?

Problem 2. Suppose a car is moving along a straight line, and its velocity $v(t)$ is given by the following graph. Draw the acceleration $a(t)$ graph.



Problem 3. Find the maximum of function $f(x) = 4x^3 - x^4$.

Problem 4. When does the function $f(x) = x^3 + 3x^2 - 9x + 1$ have the local minimum or the local maximum? Are they the "global minimum"(i.e. the usual minimum) or the global maximum (i.e. the usual maximum) as well?

Problem 5. Consider the following general form for the function (3):

$$g(x) = x^4 + cx^2 \tag{6}$$

What is the value for x when the global minimum is achieved? Show that we have to consider two cases separately (i.e. $c \ge 0$ and $c < 0$) to solve this problem, and obtain the answers for each case. Actually, in physics, our example is related to what is called "phase transition." The minimum when $c \ge 0$ and the minimum when $c < 0$ are qualitatively different. So, something strange happens when you pass through $c = 0$ as you lower or raise c. Some functions that describe phase transitions show similar behaviors. As you vary your parameters in theory something strange happens as your parameters pass through certain behavior. For example, if you cool water, something strange happens at 0 degree Celsius and it becomes ice. As another example, if you heat water, something strange happens at 100 degree Celsius and it becomes vapor. These are two examples of phase transition. It would be nice if I could talk about phase transition in later articles, but I can't as I am not an expert on it.

Summary

- "Extremum" is a collective term for maximum and minimum.
- Local extremum is an extremum around its small neighborhood.
- At a local extremum of a function, its derivative is necessarily zero.