Fundamental theorem of algebra

The fundamental theorem of algebra states that for any non-constant polynomial $f(x)$, possibly with complex coefficients, there exists at least one solution to $f(x) = 0$ where the value of x for such a solution can be possibly complex number. (A constant polynomial is a polynomial that has a constant value regardless of the value of x. In other words, a polynomial that doesn't depend on x. i.e., a simple number. For example, if $f(x) = 3$, $f(x) = 0$ doesn't have any solution.) There are several ways to prove this theorem, but we will not do so here as the proof is advanced, so we will postpone it to our later article "Proof of fundamental theorem of algebra." Instead, we will talk about its consequences in this article.

Let's say that $f(x)$ is a polynomial of degree n. According to the fundamental theorem of algebra, $f(x) = 0$ has at least one solution. Let's call such a solution $x = a_1$. Then, remembering our discussion in Problem 6. of "What is a function?," we can write $f(x) = (x - a_1)g(x)$ for a certain polynomial $g(x)$. Of course, $g(x)$ is now a polynomial of degree $n-1$ and we can apply the fundamental theorem of algebra again except for the case in which $n-1$ is 0. (In such a case, $g(x)$ is merely a number i.e., a constant polynomial.) Then, $g(x)$ has at least one solution which we call $x = a_2$. Then, we can write $g(x) = (x - a_2)h(x)$ for a certain polynomial $h(x)$. In other words, $f(x) = (x - a_1)(x - a_2)h(x)$. We can repeat this process, until the quotient polynomial becomes a constant polynomial. A constant polynomial is merely a number. Let's call the constant polynomial c. Then, we conclude

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_n)c \tag{1}$$

This factorization is unique except for the order of factors. We will prove this by showing that the assumption that the factorization is not unique leads to a contradiction. If the factorization is not unique, we can write

$$f(x) = (x - b_1)(x - b_2) \cdots (x - b_n)c \tag{2}$$

where at least one of the bs differ from all of as. Let's call such a b "b_i". From (2) we have $f(b_i) = 0$. On the other hand, from (1), we have $f(b_i) \neq 0$ as $b \neq a_1, a_2, \cdots, a_n$; you can never get zero by multiplying non-zero numbers only. This is a contradiction as $f(b_i)$ can't be zero and non-zero at the same time. We conclude our assumption was wrong; the factorization is unique.

Given this, (1) shows that nth order equation $f(x) = 0$ has generally n solutions, namely, $a_1, a_2, \cdots, a_n$. Of course, it is possible that some a_is are repeated. In this case, there are less than n solutions. The fact that nth order equation has generally n solutions will play an important role in our later article "Finding eigenvalues and eigenvectors."

Problem 1. One of the solutions for $x^3 = 15x + 4$ is $x = 4$. Find the other two solutions. (Hint[1])

Final comment. Let's consider a real-valued odd degree polynomial $f(x)$. For example, $7x^5 + 4x^4 - 3x^2 + 2x + 1$. It is odd degree because 5 is an odd number. It is a real-valued, because all the coefficients are real. An example of a polynomial which is not real-valued would be $7x^5 + (4 + i)x^4 - 3x^2 + 2x + 1$. Then, the claim is that $f(x) = 0$ has at least one real root. I will prove this in two ways.

The first way. As x approaches infinity, the highest order term can outweigh all the other terms. Thus, if the coefficient for the highest order term is positive, $f(x)$ will approach infinity, as x approaches infinity. (In our example, $f(x)$ is such a case as 7 is positive.) On the other hand, in such a case, as x approaches negative infinity, $f(x)$ will approach negative infinity. In conclusion, for a big enough x, $f(x)$ will be a positive number, while $f(x)$ will be a negative value for a small enough x. As $f(x)$ is a continuous function, between these two values of x, $f(x)$ has to cross 0. $f(x)$ cannot suddenly "jump" from a negative number to a positive number as x increases.

Problem 2. Consider the case in which the coefficient for the highest order term is negative and show that $f(x)$ has to cross 0 as well, if it is a odd degree polynomial.

Problem 3. Explain why even degree polynomial doesn't necessarily cross 0.

The second way. As $f(x)$ is real-valued, if a is a solution to $f(x) = 0$, then $\bar{a}$ must be also a solution. Thus, all the solutions must be paired. However, there are odd number of solutions according to the fundamental theorem of algebra. Therefore, not every solution can be paired, and there should be at least one solution that satisfies $a = \bar{a}$. Such a solution is real.

Summary

- The fundamental theorem of algebra states that for any non-constant polynomial $f(x)$, possibly with complex coefficients, there exists at least one solution to $f(x) = 0$ where the value of x for such a solution can be possibly complex number.

- This implies that a degree n polynomial is completely factorizable as

$$f(x) = c(x - a_1)(x - a_2) \cdots (x - a_n)$$

  for complex numbers as and this factorization is unique.

- A real-valued, odd degree polynomial has at least one solution.

[1] It implies that one of the solutions of $x^3 - 15x - 4 = 0$ is $x = 4$.