Gamma function

1 Gamma function

The gamma function $\Gamma(x)$ is defined by the following equation.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{1}$$

Problem 1. Show $\Gamma(1) = 1$.

Problem 2. Show $\Gamma(x+1) = x\Gamma(x)$. (Hint: Use integration by parts.)

Problem 3. Using the result of Problem 1 and 2, prove that $\Gamma(x) = (x-1)!$.

As advertised in our earlier article "Combination," it is worth noting that by this way the gamma function is defined for non-integers and negative numbers as well while our earlier definition for the factorial (i.e., multiplying all the natural numbers from 1 to the designated number) can be only used for the non-negative integers.

Problem 4. Show $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. (Hint: Use $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$.)

2 Euler beta function

This section closely follows the very first pages of *Superstring theory* by M.B.Green, J.H.Schwarz and E.Witten where the authors deal with the inception of string theory. The Euler beta function, which we will now introduce, played a pivotal role in the advent of string theory.

The Euler beta function is defined by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{2}$$

Problem 5. Show the following:

$$B(x-1, y+1) = \frac{y}{x-1} B(x,y), \qquad B(x+1, y-1) = \frac{x}{y-1} B(x,y) \tag{3}$$

Problem 6. Show the following:

$$B(x+1, y) = B(x,y) - B(x, y+1) \tag{4}$$

Now, let's consider the following function:

$$C(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \tag{5}$$

Problem 7. Show the following: (Hint[1])

$$C(x-1, y+1) = \frac{y}{x-1} C(x, y), \qquad C(x+1, y-1) = \frac{x}{y-1} C(x, y) \tag{6}$$

Problem 8. Show the following: (Hint[2])

$$C(x+1, y) = C(x, y) - C(x, y+1) \tag{7}$$

Problem 9. Check $C(1, 1) = B(1, 1)$.

As $C(x, y)$ coincides with $B(x, y)$ when $x = y = 1$ and has the same recursion relation with $B(x, y)$, we can suspect that $C(x, y)$ is actually $B(x, y)$. Indeed, one can rigorously prove that this is the case using several more relations. In other words,

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{8}$$

It is remarkable that Euler had studied the Euler beta function two centuries before Veneziano used it for physics in 1968. It often happens that physicists find use of old works of mathematicians who perhaps never dreamed that their work would be useful for physics someday.

Summary

- $\Gamma(x) = (x-1)!$
- While factorial is defined only for positive integers, gamma function is defined for all numbers.

[1] Use integration by parts.

[2] Use $t^x(1-t)^{y-1} = t^{x-1}(1-t)^{y-1} - t^{x-1}(1-t)^y$.