Geodesics in the presence of constant gravitational field

In earlier article "Fermat's principle and the consistency of physics," I explained that the light always takes the path that extremizes the time it took. This is also the case for general relativity. In general relativity, a particle always takes the path that extremizes the proper time it took. This path is called "geodesics." After all, proper time is the time measured by the particle. In this article, at least in a suitable limit (i.e. small velocity and weak gravitational field), we will show that this consideration leads to the extremization of non-relativistic Lagrangian of particles moving. To this end, let's calculate the proper time of particles moving with velocity v. We have:

$$(c\delta\tau)^2 = (c\delta t)^2 - (dx^2 + dy^2 + dz^2) = (c\delta t)^2 - (v\delta t)^2 \tag{1}$$

$$\delta\tau = \sqrt{1 - v^2/c^2}\,\delta t \tag{2}$$

This relation makes sense, if you think about time dilation. Time goes more slowly for moving particle than the one for outside observer. Indeed $\delta\tau$ is smaller than δt. When v is much smaller than c, this amounts to:

$$\delta\tau \approx \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)\delta t \tag{3}$$

Now, recall that we had the following formula in our earlier article "By how much does time go more slowly at a lower place."

$$\Delta t_A \approx \left(1 + \frac{gh}{c^2}\right)\Delta t_B \tag{4}$$

Applying this formula to (3), we get:

$$\delta\tau \approx \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)\left(1 + \frac{gh}{c^2}\right)\delta t \tag{5}$$

which implies, for $gh \ll c^2$,

$$\delta\tau \approx \left(1 - \frac{1}{2}\frac{v^2}{c^2} + \frac{gh}{c^2}\right)\delta t \tag{6}$$

So, if we call the thing in the parenthesis "L_0" this is the one that is extremized. Therefore, we have the following Euler-Lagrange equation:

$$\frac{\partial L_0}{\partial q} - \frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{q}}\right) = 0 \tag{7}$$

However, the non-relativistic Lagrangian of such a particle is given by following:

$$L = \frac{1}{2}mv^2 - mgh \tag{8}$$

since this is the kinetic energy subtracted by potential energy in a constant gravitational field. Therefore, we can re-express L_0 as follows:

$$L_0 = 1 - \frac{m}{c^2}L \tag{9}$$

Plugging this back to (7) we have:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{10}$$

So, we recover the Euler-Lagrange equation for moving particle in a constant gravitational field! The extremization of proper time indeed implies the extremization of Lagrangian.

Problem 1. This actually suggests that the relativistic Lagrangian is given by the proper time up to a multiplicative factor. Recalling that the dimension of Lagrangian is that of energy, it makes sense if this multiplicative factor is mc^2. In other words, the action is given by

$$S = \int mc^2 d\tau \tag{11}$$

In this problem, we will consider the case in which there is no gravitational field. Then, the Lagrangian is given by

$$L = mc^2\sqrt{1 - \frac{v^2}{c^2}} \tag{12}$$

Show that the conjugate momentum of the above Lagrangian is the relativistic momentum of a particle with mass m, and the Hamiltonian of the above Lagrangian is the relativistic energy of a particle with mass m.

Problem 2. In "Einstein summation convention," we have seen that a vector can be represented by an index. For example, $\vec{v}$ can be represented by v^a, where $v^1 = v_x$, $v^2 = v_y$, $v^3 = v_z$. A 4-vector can be also represented by an index, but the index runs from 0 to 3, instead of 1 to 3, as the 0th component denotes the time component. For example, the 4-vector x^a, which denotes the time and the position, means $x^0 = t, x^1 = x, x^2 = y, x^3 = z$, if we the natural unit $c = 1$. Similarly, the 4-momentum p^a is given by $p^0 = E, p^1 = p_x, p^2 = p_y, p^3 = p_z$, again using the natural unit. Given this, show that

$$p^a = m\frac{dx^a}{d\tau} \tag{13}$$

Summary

- In general relativity, a particle always takes the path that extremizes the proper time it took.
- In Newtonian limit, one can easily show that such a path extremizes the Newtonian Lagrangian, $L = T - V$.
- In the absence of gravitational field, the relativistic action is given by $S = \int mc^2 d\tau$, which implies that the relativistic Lagrangian is given by $L = mc^2\sqrt{1 - v^2/c^2}$.

- The 4-momentum is given by

$$p^a = m\frac{dx^a}{d\tau}$$