Harmonic oscillator and circular motion

As advertised in the last article, we will derive the period of harmonic oscillator with a spring constant k and mass m. Before doing so, we need to closely look at some properties of a circular motion. In "Centripetal force," we have seen that an object moving with speed v in a circle with radius r is receiving centripetal force of $F = mv^2/r$ toward the center of the circle. Also, in "Rotation," we have seen that the rotation speed is given by $v = r\omega$, where ω is angular velocity. Then, the centripetal force can be re-expressed as

$$F = mr\omega^2 \tag{1}$$

We will make use of this relation soon.

See Fig. 1 for the circular motion. The center of the circle is at the origin. Let's denote the position of the orbiting object by a vector $\vec{r} = xi + yj$. In other words, (x, y) is the coordinate of the object. Of course, if we denote the angle $\vec{r}$ makes with x-axis by angle θ as in Fig. 1. We have

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{2}$$

It goes without saying that

$$|\vec{r}| = \sqrt{x^2 + y^2} = r \tag{3}$$

Given this, what is the force exerted upon the orbiting object? Of course, it is given by (1), but what is missing is its direction. (1) is just the magnitude of the force. Remembering that the centripetal force is toward the origin, the centripetal force must be the opposite direction

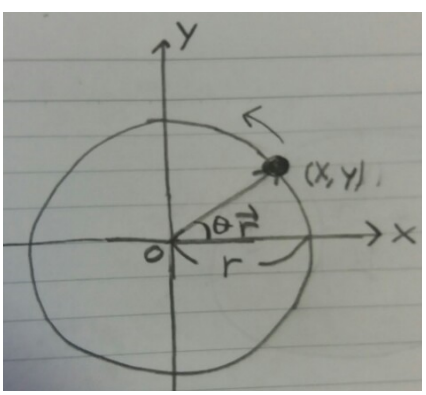

Figure 1: a circular motion

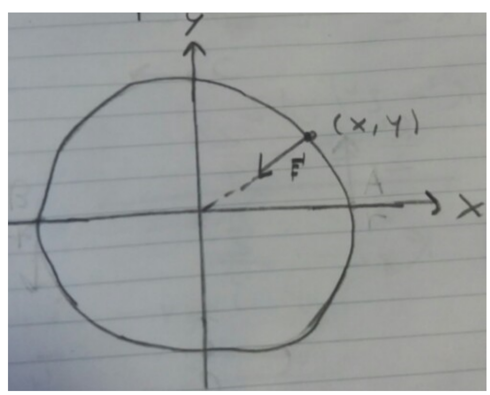

Figure 2: the force acting on circularly orbiting object

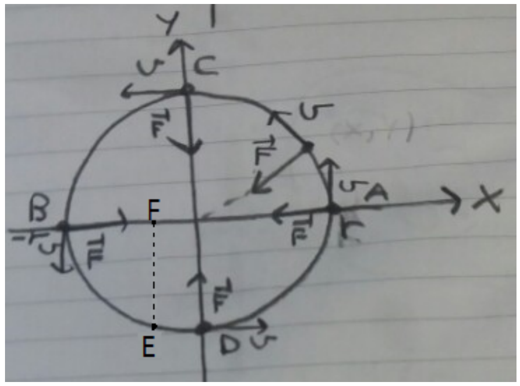

Figure 3: a circular motion and a harmonic oscillator

of $\vec{r}$. See Fig. 2. Therefore, the force must be given by something multiplied by $-\vec{r}$. The correct answer is

$$\vec{F} = -m\omega^2\vec{r} \tag{4}$$

Check that $|\vec{F}| = mr\omega^2$ as in (1). Now, let's re-express (4). If we denote the x-component of $\vec{F}$ by F_x and the y-component of $\vec{F}$ by F_y, we have

$$\begin{aligned} \vec{F} &= -m\omega^2\vec{r} && (5)\\ F_x i + F_y j &= -m\omega^2(xi + yj) && (6)\\ F_x &= -m\omega^2 x && (7)\\ F_y &= -m\omega^2 y && (8) \end{aligned}$$

Thus, we see that F_x is the function of x only while F_y is the function of y only. So, they can be regarded as if they don't "talk" to each other. They are completely decoupled. Therefore, if we had an object of which the x-component of the force it receives is given by (7), the x-coordinate of the object must be necessarily the x-coordinate of the circular motion.

Actually, there is such an object! An object connected to spring. The object receives $F_x = -kx$. Comparing with (7), we see that

$$\omega = \sqrt{\frac{k}{m}} \tag{9}$$

As the period of a circular motion is given by $T = 2\pi/\omega$, we get

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{10}$$

This completes the proof. In our later article "Harmonic oscillator," we will derive this formula again using calculus.

Final comment, let's actually think qualitatively whether our picture in this article is correct. See Fig. 3. The projection of the circular motion to x-axis is the motion of harmonic oscillator. For example, the position E in circular motion corresponds to the position F

in harmonic oscillator. The equilibrium is at $x = 0$ and the harmonic oscillator oscillates between $x = -r$ and $x = r$. When the object orbiting is at position A or B, the x-component of its velocity is zero; its velocity has only y-component. This is consistent with the fact that when an object in harmonic oscillator is farthest from the equilibrium, it momentarily stops; its velocity is zero. When the object orbiting is at position C or D, the x-component of the centripetal force is zero. This is consistent with the fact that an object in harmonic oscillator doesn't receive any force when it's passing equilibrium (i.e. $x = 0$). Also, you see that at such points (C and D) the x-component of velocity is greatest. It is given by $r\omega$.

Problem 1. Let's consider an object with mass m is connected to a spring with spring constant k. If it oscillates with the amplitude r, find its speed when it passes the equilibrium (i.e. the maximum speed).

Summary

- The position of an object in harmonic oscillator is given by x-coordinate of an object rotating.
- We have
$$\omega = \sqrt{\frac{k}{m}}$$
which implies that the period is given by
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$