Harmonic oscillator

Harmonic oscillator is a system in which an object receives a force proportional and opposite to the displacement from equilibrium. In other words,

$$\vec{F} = -k\vec{x} \tag{1}$$

We have learned this in our earlier article "Hooke's law and harmonic oscillator." In this article, as advertised, we will calculate the period of harmonic oscillator using calculus.

First of all, from (1), we have:

$$m\frac{d^2x}{dt^2} = -kx \tag{2}$$

This is a differential equation. We have to find the solution to this equation. The solution is given by sine and cosine functions. For example, if we try:

$$x = A\cos\omega t + B\sin\omega t \tag{3}$$

Then, (2) becomes:

$$m\left(-A\omega^2\cos\omega t - B\omega^2\sin\omega t\right) = -k\left(A\cos\omega t + B\sin\omega t\right) \tag{4}$$

Therefore, we have:

$$\omega = \sqrt{\frac{k}{m}} \tag{5}$$

Therefore, the full solution is:

$$x = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t \tag{6}$$

Here, A and B are arbitrary constants that can be determined from the initial conditions. Also, using the addition rule of sine function, one can also show that the above equation can be re-written as following:

$$x = D\sin(\sqrt{\frac{k}{m}}t + \phi_0) \tag{7}$$

where $\tan\phi_0 = A/B$, $D = \sqrt{A^2 + B^2}$.

Either way, we indeed see that the object attached to the spring oscillates. In particular, the period, T can be obtained as follows:

$$\sqrt{\frac{k}{m}}T = 2\pi \tag{8}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{9}$$

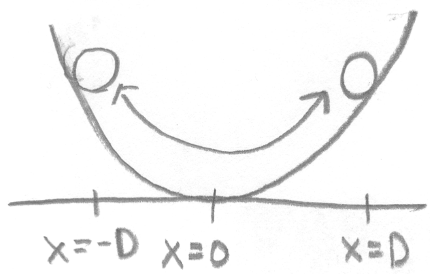

Figure 1: Potential energy of harmonic oscillator

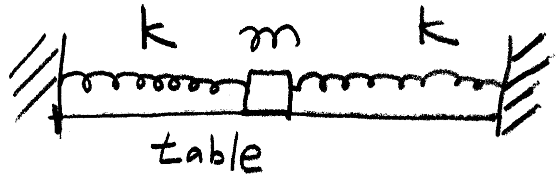

Figure 2: an object with mass m attached to two springs with each spring constant k

Given this, we can look at this problem slightly in a different angle using energy. What would be the potential energy of spring? We have:

$$U(x) = -\int F(x)dx = -\int -kxdx = \frac{1}{2}kx^2 \tag{10}$$

Now, it is an easy exercise to check that the total energy is conserved. From (7), the velocity is given by:

$$v = \frac{dx}{dt} = D\sqrt{\frac{k}{m}}\cos(\sqrt{\frac{k}{m}}t + \phi_0) \tag{11}$$

Then, the total energy is given by:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kD^2\left(\cos^2(\sqrt{\frac{k}{m}}t + \phi_0) + \sin^2(\sqrt{\frac{k}{m}}t + \phi_0)\right) = \frac{1}{2}kD^2 \tag{12}$$

which is indeed constant.

Let's examine this further. The first term in the above equation is kinetic energy while the second term is potential energy. As their sum is constant, the kinetic energy is maximum when the potential energy is minimum, which, in our case, is zero. This implies we have $x = 0$ in such a case. In other words, the speed of the object is greatest when $x = 0$, as we figured out in our earlier article "Hooke's law and harmonic oscillator." Similarly, the potential energy is maximum, when the kinetic energy is minimum, which in our case is zero. In such a case $x = D$ or $x = -D$ as $\frac{1}{2}kx^2$ must be equal to $\frac{1}{2}kD^2$. In other words, the object momentarily stops (i.e. $v = 0$)when it is maximally stretched or maximally compressed.

All this can be more visualized if we draw the potential energy as we did in Figure 1 of our last article "Kinetic energy and Potential energy in one dimension." See Fig.1. If you place a ball at $x = D$ or $x = -D$, it will oscillate between these two points.

Actually, we can go further. Remember that in our last article, I explained that an object placed near $x = a$ will oscillate. We can actually calculate the period of this oscillation assuming that the object was placed sufficiently near $x = a$. Let us show you how one can do this. First, we Taylor-expand $U(x)$ around $x = a$ as follows:

$$U(x) = U(a) + U'(a)(x-a) + \frac{1}{2}U''(a)(x-a)^2 + \frac{1}{6}U'''(a)(x-a)^3 + \cdots \tag{13}$$

Notice that the second term is absent since $U'(a) = 0$ as $x = a$ is local minimum. Now, if x is sufficiently close to a, the term proportional to $(x-a)^3$ and higher order terms can be ignored. Moreover, the value of $U(a)$ is not important for our purpose, as it doesn't affect the dynamics. The whole "terrain" will move simply upward or downward. What is important is the relative height of a place on the terrain, not the overall height. This can be easily seen as $U(a)$, a constant, vanishes if we calculate the force by $F = -\partial U/\partial x$ by plugging in $U(x)$ from the above equation. In any case, $U(x)$ for x near a is given by:

$$U(x) \approx U(a) + \frac{1}{2}U''(a)(x-a)^2 \tag{14}$$

and if we identify $U''(a)$ with k, the period of an object oscillating around $x = a$ is given by

$$T = 2\pi\sqrt{\frac{m}{U''(a)}} \tag{15}$$

This is the power of harmonic oscillator. The potential around stable equilibrium can be approximated by harmonic oscillator, and the dynamics can be simply analyzed, albeit through approximation. Calling black holes the harmonic oscillators of 21st century, Prof. Andrew Strominger once remarked "In the twentieth century, many problems across all of physics were solved by perturbative methods which reduced them to harmonic oscillators." Now, you understand what he meant, even though you may not be sure about the black hole part.

Problem 1. Let's say that the point $x = a$ is an equilibrium. In other words, if U is the potential, $U'(a) = 0$. Given this, relate the sign of $U''(a)$ with the equilibrium being stable or unstable. (You don't need to consider the case $U''(a) = 0$.)

Problem 2. See Fig.2. An object with math m is on a table and attached to two springs with each spring constant k. What will be the period of the object's oscillation due to the springs? Let's assume that the friction due to the table is negligible. (Hint[1])

Summary

- The solution to $m\frac{d^2x}{dt^2} = -kx$ is given by

  $$x = A\cos\omega t + B\sin\omega t = D\sin(\omega t + \phi_0)$$

  where $\omega = \sqrt{\frac{k}{m}}$.
- $\omega T = 2\pi$ where T is the period.
- The potential energy of spring is given by $\frac{1}{2}kx^2$.
- We can analyze an oscillation near a stable equilibrium by Taylor-expanding the potential energy as follows:

  $$U(x) = U(a) + U'(a)(x-a) + \frac{1}{2}U''(a)(x-a)^2 + \cdots$$

  Then, we have $U'(a) = 0$, and $U''(a)$ can be regarded as the spring constant.

[1] Try to think about what the force will be when the object is distance x away from the equilibrium.