Homology

In the last article, we introduced de Rham cohomology. There is a way of denoting the same information that de Rham cohomology carries without using the language of differential forms. It is homology.

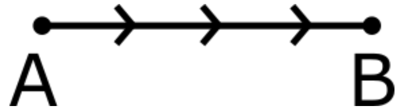

Figure 1: An oriented straight line with its boundary, $B - A$

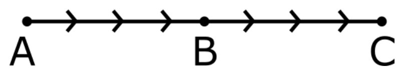

Figure 2: Two oriented straight lines with their boundaries. $B - A$ corresponds to the first one, while $C - B$ to the second one.

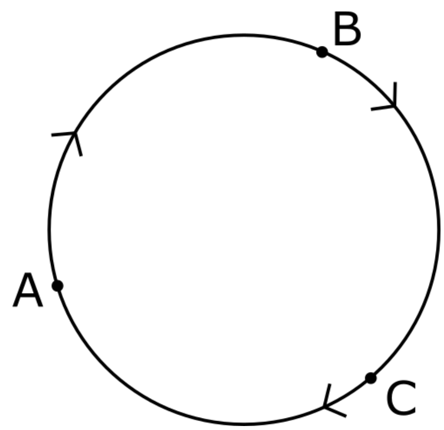

Figure 3: Circle ABC and its orientation. It has no boundary.

To understand homology, we need to first start with a boundary operator, just like we needed to understand exterior derivative first before understanding de Rham cohomology. So what is a boundary operator? It is the same boundary operator that appears in Green's theorem and Stokes' theorem. For example, see Fig.1. The boundary of simple line drawn in the figure is two points, A and B. However, we need to assign a orientation in the line. In the figure, the orientation is denoted by the arrows. We will see the usefulness of assigning a orientation, soon. If we call the line drawn in the figure $\overline{AB}$, its boundary $\partial\overline{AB}$ is given by

$$\partial\overline{AB} = B - A \tag{1}$$

Notice that there is a negative sign in front of A. Think it along this way. Recall the generalized Stokes' theorem

$$\int_M df = \int_{\partial M} f \tag{2}$$

In our case, we have

$$\int_{\overline{AB}} df = \int_{\partial\overline{AB}} f \tag{3}$$

which is the simple fundamental theorem of calculus, namely,

$$\int_A^B df = f(B) - f(A) \tag{4}$$

From (3) to (4), we plugged in (1).

Of course, if the orientation of the line were not from A to B but from B to A, its boundary would have been $A - B$.

To understand the usefulness of orientation. See Fig. 2. We see that

$$\overline{AC} = \overline{AB} + \overline{BC} \tag{5}$$

What is the boundary of $\overline{AC}$? It is given by

$$\partial \overline{AC} = C - A \tag{6}$$

What is the boundary of $\overline{BC}$? It is given by

$$\partial \overline{BC} = C - B \tag{7}$$

Now, notice that assigning an orientation really makes sense. If we apply boundary operator to (5) we have

$$\partial \overline{AC} = \partial \overline{AB} + \partial \overline{BC} \tag{8}$$

$$C - A = (B - A) + (C - B) \tag{9}$$

which makes sense. It shows that ∂ is a genuine linear operator. We would have never had such a relation if $\partial \overline{AB}$ were defined by $A + B$ instead of $-A + B$.

We can go further. See Fig. 3. The circle ABC, which is a sum of $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ has no boundary, as

$$\partial \overline{AB} + \partial \overline{BC} + \partial \overline{CA} = (B - A) + (C - B) + (A - C) = 0 \tag{10}$$

Now, let's consider the boundary of a 2-dimensional manifold. In Fig. 4, you see the disk (i.e., the interior of a circle). Its boundary is given by the circle D of which the orientation is drawn in the figure. In Fig. 5, you see another disk. Its boundary is given by the circle E. Now, see Fig. 6. You see an object which is given by the disk in Fig. 4 subtracted by the disk in Fig. 5? What is its boundary? It is $D - E$, as it must be the boundary of the disk in Fig. 4 subtracted by the boundary of the disk in Fig. 5. (Remember that a boundary operator is a linear operator.) Notice that the outer boundary has the same orientation as the one in Fig. 4, while the inner boundary has the opposite orientation to the one in Fig. 5, as we have $-E$ instead of E.

We can define the boundary operator of higher-dimensional object similarly. An important property of a boundary operator is $\partial^2 = 0$. In other words, a boundary of a boundary is always zero. For example, we have already seen that a circle, which is a boundary of disk,

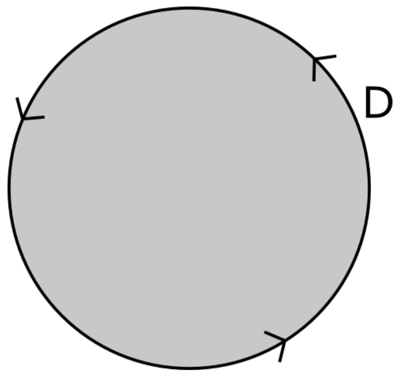

Figure 4: Disk with its boundary given by circle D.

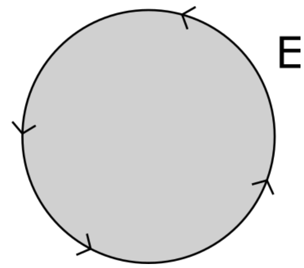

Figure 5: Disk with its boundary given by circle E.

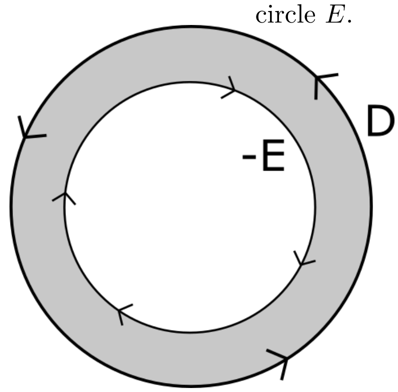

Figure 6: Surface as the result of subtracting disk in Fig. 5 from that in Fig. 4. Its boundary is $D - E$.

does not have a boundary. Similarly, a 2-sphere, which is a boundary of a 3-ball, does not have a boundary.

Remember that we had $d^2 = 0$ for exterior derivative. There, we saw that a differential form q which can be expressed as $q = dr$ always satisfied $dq = 0$ because $d^2 = 0$

Similarly, in case of the boundary operator, we see that an object which is a boundary of another object has no boundary. An r-dimensional object a that is a boundary of another object (i.e. $a = \partial b$) is called r-boundary. An r-dimensional object g that has no boundary (i.e. $0 = \partial g$) is called r-cycle. Thus, an r-boundary is always r-cycle, but not necessarily vice versa, just as an exact form is always closed but not necessarily vice versa.

The element of the rth de Rham cohomology group is given by the r-form which is closed, but not exact. Similarly, the element of the rth homology group is given by the r-cycle which is not a boundary.

Now, let's formally define the homology group. Let K be a simplicial complex. Here, simplicial complex is the vector space where higher dimensional objects that we mentioned, such as lines, spheres, and disks live. Then, the rth cohomology group $H_r(K)$ is defined by the following equivalence relation.

$$c \sim c + \partial d \tag{11}$$

where c is an r-cycle and d is an $(r+1)$-chain. (The $(r+1)$-chain is the correct and precise mathematical terminology, but you can roughly think it as $(r+1)$-dimensional object.)

Now, let's check that it carries the same information as de Rham cohomology. In particular let's check that the 1st homology group of torus carries the same information as the 1st de Rham cohomology group of torus.

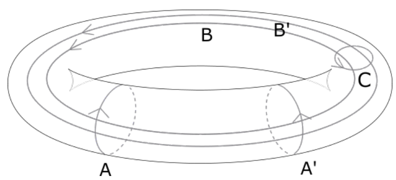

Figure 7: Schematic representation of a torus. Circles A, A', B and B' represent examples of the 1st homology group as they are 1-cycles.

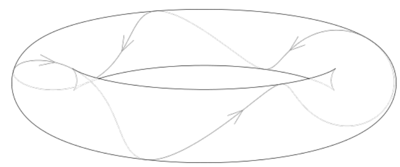

Figure 8: Schematic representation of a torus with an homology element given by $4A + B$. See that this element winds the A-cycle 4 times while it winds only one time the B-cycle.

See Fig. 7. the circle A, the circle A' and the circle B, the circle B' are examples of elements of the 1st homology group, as they are 1-cycles (i.e., they are 1-dimensional and don't have boundaries) but are not boundaries of any 2-dimensional objects in torus. (It may seem that they are boundaries of 2-dimensional objects, but remember: in mathematics, torus is the 2-dimensional surface, not the 3-dimensional interior) Also, notice that loop C is a boundary, so it doesn't count as a homology element.

Now, notice that A and A' are the same homology element, because if we call the band enclosed by A and A', by D, we have

$$\partial D = A' - A \tag{12}$$

which implies

$$A \sim A + \partial D = A' \tag{13}$$

Similarly, B and B' are the same homology element. Of course, A and B are not the only homology elements. In Fig. 8, you see a homology element which is given by $4A + B$. It winds the A-cycle 4 times while it winds the B-cycle 1 time.

In conclusion, the 1st homology group of torus is given by $H_1(\text{Torus}) = \mathbb{Z}^2$, where $\mathbb{Z}$ denotes an integer. In other words, an element in the 1st homology group of a torus can be uniquely specified by two integers, namely, how many times you wrap A-cycle and how many times you wrap B-cycle. If you wrap them in the other way around, you get negative integers. So, the information about the 1st homology group of torus is in the number 2, the exponent of $\mathbb{Z}^2$. Remember that we also had 2 in case of de Rham cohomology group, namely, $H^1_{dR}(\text{Torus}) = \mathbb{R}^2$. According to de Rham theorem, these two numbers in the homology group and the de Rham cohomology group always match. We will sketch a proof of this theorem in the next article.

Problem 1. What is $H_1(S^1 \times S^1 \times S^1)$?

Summary

- In the case of homology, the natural object is an n-dimensional object while the natural objects are differential forms in case of de Rham cohomology.
- ∂ is called the "boundary operator."
- A cycle f satisfies $\partial f = 0$. A boundary f satisfies $f = \partial g$ for some manifold g.
- Homology tells us how many more cycles there are than boundaries.
- An r-cycle is a r-chain that has no boundary. i.e., c such that

$$\partial c = 0$$

- An r-boundary is a r-chain that is a boundary of an $(r+1)$-chain. i.e., c such that

$$c = \partial d$$

  where d is an $(r+1)$-chain.
- The rth homology group $H_r(K)$ is an equivalence class defined by

$$c \sim c + \partial d$$

  where c is an r-cycle and d is an $(r+1)$-chain.