The imagination in mathematics: “Desargues’s theorem”

It is my impression that in popular belief, mathematicians and theoretical physicists are engaged in nothing more than very complicated and boring logical thinking. Indeed, I should confess that I myself held similar views when I was young. I thought that what theoretical physicists do was only matter of finding tricks to solve complicated differential equations. However, I have since discovered that such views are wrong, and I find it unfortunate that many people continue to hold them.

That these views are wrong has been clearly expressed by several renowned mathematicians and theoretical physicists. Edward Witten, the most renowned string theorist and winner of the Fields Medal (the mathematical equivalent of the Nobel Prize), said:

“Most people who haven’t been trained in physics probably think of what physicists do as a question of incredibly complicated calculations, but that’s not really the essence of it. The essence of it is that physics is about concepts, wanting to understand the concepts, the principles by which the world works.”

In “the Evolution of Physics,” Albert Einstein and Leopold Infeld wrote:

“The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill. To raise new questions, new possibilities, to regard old problems from a new angle, requires creative imagination and marks real advance in science.” On another occasion, Albert Einstein said: “Logic will get you from A to B. Imagination will take you everywhere.”

Einstein is telling us that theoretical physicists work by playing around with concepts and imagining them, rather than applying pure logic and calculation with no sense of what is actually going on behind the formulas. Of course, physicists do still need, eventually, to fill the gaps with logic and calculation but that doesn’t diminish the importance of imagination in math and theoretical physics.

In this article, I will give you an example of how “imagination” can be useful in mathematics.

See Fig. 1. Desargues’s theorem says that, if $\overline{Aa}$, $\overline{Bb}$, $\overline{Cc}$ meet at a point (denoted by O), then the point in which $\overline{AB}$ and $\overline{ab}$ meet, the point in which $\overline{BC}$ and $\overline{bc}$ meet, and the point in which $\overline{AC}$ and $\overline{ac}$ meet, if they exist, all lie on the same line (denoted by the red line).

Desargues’s theorem is a theorem on two triangles on 2-dimensional plane. There is nothing 3-dimensional about this theorem. However, if we see Fig. 1 3-dimensionally by using our imagination, it is easy to prove Desargues’s theorem. First, regard $OABC$ and $Oabc$ as two triangular cones. Their bases are $\triangle ABC$ and $\triangle abc$. Then, the plane that contains the former and the plane that contains the latter must meet, because of our assumption that the

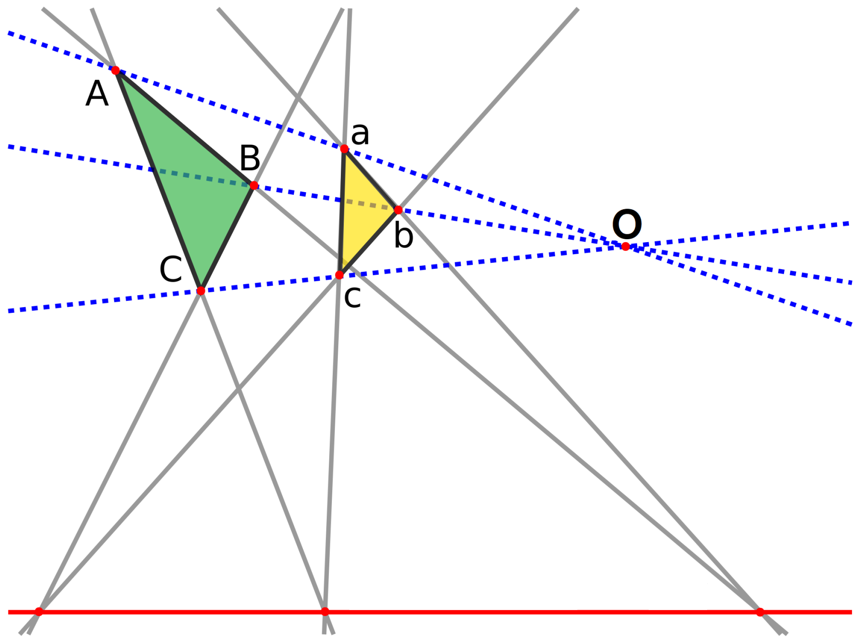

Figure 1: Desargues's theorem. The plane that contains $\triangle ABC$ and the plane that contains $\triangle abc$ meet at the red line. [1]

three intersection points exist. Given this, think about the fact that the intersection of two planes, if they indeed intersect, is always a straight line. Now, it is easy to see that these two planes intersect in the red line in the figure.

Usually, one thinks that it complicates the problem, if we upgrade 2-dimension to 3-dimension. However, in this case, it simplified the problem. With some imagination, the theorem is easy to prove. Of course, there are harder ways to prove Desargues's theorem.

References

[1] Adapted from `https://commons.wikimedia.org/wiki/File:Desargues_theorem_alt.svg`