# Motions on an inclined plane

See Fig. 1. An object with mass $m$ is on an inclined plane with slope $\theta$ degrees. It is initially at rest at the point $A$. Assuming that there is no friction, what is its speed after it has slided down along the plane $s$ meters? (i.e. when the object reaches the point $B$.) We will solve this problem using two different methods.

The first method. Notice that the gravitational force acting on the object can be expressed as the sum of the force normal (i.e. perpendicular) to the plane and the force along the plane. From the figure, we can easily see that their magnitudes are $mg\cos\theta$ and $mg\sin\theta$ respectively. Now, notice that the object moves only along the plane; it doesn't move at all along the perpendicular direction of the plane. It means that the force perpendicular to the plane is exactly canceled by the normal force introduced in the last article. In this case, the magnitude of the normal force is $mg\cos\theta$. Given this, the only force left to be considered is the component of the gravitational force along the plane. Its magnitude is $mg\sin\theta$. Therefore, the acceleration of the object is given by $g\sin\theta$ from Newton's second law. Since it slides $s$ meters, we can easily obtain the speed. Recall the following formula from our earlier article "Constant acceleration in 1-dimension."

$$s = \frac{v_f^2 - v_i^2}{2a} \tag{1}$$

Plugging $v_i = 0$ and $a = g\sin\theta$, we conclude that the final speed is given by $\sqrt{2gs\sin\theta}$.

The second method. There are two forces acting on the object. The gravitational force and the normal force. Therefore, the gain of kinetic energy of the object is given by the sum of the work done by the gravitational force, and the sum of the work done by the normal force. However, the object moves along the direction perpendicular to the normal

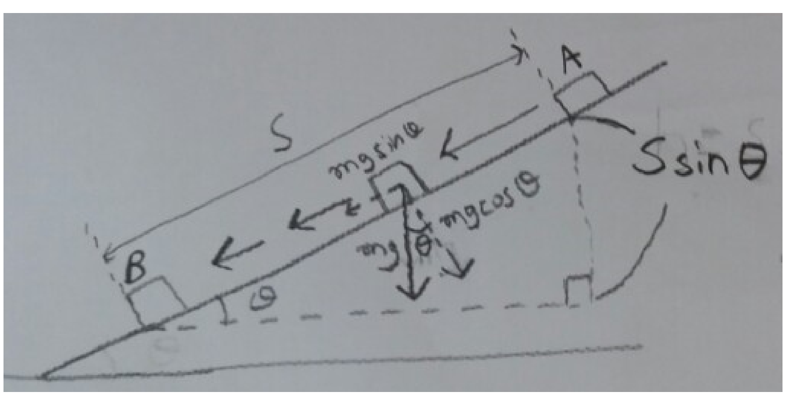


Figure 1: an object sliding without friction

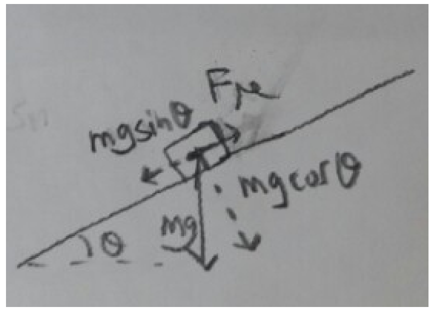


Figure 2: an object sliding with friction

force, as it is moving along the plane. Therefore, the work done by the normal force is zero. Therefore, the gain of the kinetic energy of the object is given solely by the work done by the gravitational force. Thus, we can use the conservation of energy introduced in "Potential energy and conservation of energy." If we denote the height of point $A$ by $h_A$ and the height of point $B$ by $h_B$ we have

$$\frac{1}{2}mv_i^2 + mgh_A = \frac{1}{2}mv_f^2 + mgh_B \tag{2}$$

As we can see $h_A = h_B + s\sin\theta$ from the figure, we obtain

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = mg(h_A - h_B) = mgs\sin\theta \tag{3}$$

Plugging $v_i = 0$, we conclude again that the final speed is given by $\sqrt{2gs\sin\theta}$.

**Problem 1.** In our case, how long does it take for the object to slide down $s$ meters from $A$ to $B$? Notice that your answer must be infinite when $\theta = 0$, since the object hardly moves at all in such a case. Notice also that when $\theta = 90°$, your answer must be equal to the answer in the case of free fall, since in such a case, the object is hardly touching the plane at all and just falls down. Check that your answer correctly reproduces the correct answer for these two cases.

Now, let's consider the case when there is a constant friction $F_\mu$ when the object slides down. See Fig. 2. We assume that all the other conditions are same. Then, what would be the final speed? We know that the kinetic energy of the object can be expressed as follows:

$$\begin{aligned}\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \quad = \quad & \text{the work done by the gravitational force} \\ & + \text{ the work done by the normal force} \\ & + \text{ the work done by the friction}\end{aligned} \tag{4}$$

We know that the work done by the gravitational force is given by $mg(h_A - h_B) = mgs\sin\theta$. We know that the work done by the normal force is 0. We know that the work done by the friction is $F_\mu s\cos 180° = -F_\mu s$. Here $180°$ is the angle between the force and the direction of motion. As the friction is exerted on the object in the opposite direction of its moving direction, the friction is doing negative work. Plugging everything in, if the initial speed is 0, the final speed is given by (**Problem 2.** Check!)

$$v_f = \sqrt{2\left(g\sin\theta - \frac{F_\mu}{m}\right)s} \tag{5}$$

**Problem 3.** Obtain the above result again, using the Newton's second law (i.e. the "first method" we used for our earlier case without friction).

Let us conclude this article with a comment. Let's re-write (4). We have

$$\left(\frac{1}{2}mv_f^2 + mgh_B\right) = \left(\frac{1}{2}mv_i^2 + mgh_A\right) - F_\mu s \tag{6}$$

According to our earlier article "Potential energy and conservation of energy" the terms in the above parenthesis are called energy, and they are supposed to be conserved (i.e. the term in the parentheses in the left-side is supposed to be equal to the term in the parenthesis in the right-side). But, now, we apparently see that the energy is not conserved, as there is an extra term on the right-hand side, namely $-F\mu s$. We can interpret this as a part of energy was converted into heat and sound due to friction. The amount of energy so converted is equal to $F_\mu s$. In other words, if we re-write the above equation as follows,

$$\left(\frac{1}{2}mv_f^2 + mgh_B\right) + F_\mu s = \left(\frac{1}{2}mv_i^2 + mgh_A\right) \tag{7}$$

We see that the initial energy is equal to the sum of the final energy carried by the object and the heat energy produced due to friction. In conclusion, by including the heat energy, we can make the law of the energy conservation survive.

## Summary

- If you place an object on an inclined plane, and if it slides, you can obtain its speed using the conservation of energy, as normal force does not do work. (assuming no friction)