Proof by induction

Consider the following statement:

"$4^n - 1$ is divisible by 3 for any positive integer n."

Is this true? Let's check. For $n = 1$, we have $4^1 - 1 = 3$ which is divisible by 3. For $n = 2$, we have $4^2 - 1 = 15$ which is also divisible by 3. For $n = 3$, we have $4^3 - 1 = 63$ which is also divisible by 3. And so on. Therefore, we can see that the statement seems plausible; we have no evidence that the statement is wrong as all the ns we have tried satisfied the statement.

We might be able to check whether the statement is true by this way, but it would take infinite time as there are an infinite number of positive integers. Therefore, it is not a good idea; no matter how many ns you try, there will be still infinitely many ns that still need to be checked. Moreover, there is no guarantee that the statement is always satisfied for some n you didn't try. However, there exists a good method to prove that the statement is true. We call this method "proof by induction," which we will now explain.

The first step of the method is checking that the statement is true for $n = 1$. We have already done it, as $4 - 1 = 3$ is divisible by 3.

The second step of the method is checking that the statement is true for $n + 1$ if the statement is true for n. Let's check this. If the statement is true for n, we can write $4^n = 3k + 1$ for some positive integer k. Then, we have

$$4^{n+1} - 1 = 4 \times 4^n - 1 = 4 \times (3k + 1) - 1 = 4 \times (3k) + 4 - 1 = 3 \times (4k + 1) \tag{1}$$

As $4k + 1$ is a positive integer if k is a positive integer, we indeed showed that the statement was satisfied for $n + 1$ if it is satisfied for n.

Now, let's see what kind of the consequences the second step has. Since the statement is satisfied for $n = 1$, the statement must be also satisfied for $n = 1 + 1 = 2$. Moreover, the statement must also be satisfied for $n = 2 + 1 = 3$ as the statement is now satisfied for $n = 2$. Furthermore, the statement must also be satisfied for $n = 3 + 1$ as the statement is now satisfied for $n = 3$. Continuing this way, the statement must be satisfied for $n = 4, 5, 6, 7 \cdots$ and eventually for any positive integers; no matter how big n you give me, say, $n = 10000000$, I can show you that the statement is true, since the statement is true for $n = 1, 2, 3, \cdots, 9999999, 10000000, \cdots$. This is the power of induction. Of course, not all the kind of statements that we have considered in this article can be proved by induction, but many can.

Problem 1. Prove the following:

$$\sum_{s=1}^{n} s^2 = \frac{n(n+1)(2n+1)}{6} \tag{2}$$

(Hint.[1]) A comment. Even though we can prove the above statement by induction, the method of induction doesn't give you what the sum must be, if you haven't correctly guessed it at the first place by another method. For example, if somebody asks you to express $\sum_{s=1}^{n} s^3$ by a polynomial of n, the method of induction won't give you the answer. Nevertheless, if you guess the correct answer by another method, the method of induction will enable you to check that the answer so obtained is absolutely correct.

Summary

- The first step for proof of induction is checking that the statement is true for $n = 1$.
- The second and final step for proof of induction is checking that, if the statement is true for n, the statement is true for $n + 1$ as well.

[1] Use $\sum_{s=1}^{m+1} s^2 = \left(\sum_{s=1}^{m} s^2\right) + (m+1)^2$.