Infinite-dimensional vector

So far, we have only considered vectors in finite-dimensional vector space. However, we will see that we need to consider vectors in infinite-dimensional vector space in quantum mechanics. Therefore, we will introduce infinite-dimensional vectors in this article.

Recall how a vector in a finite-dimensional vector space can be represented. We need n numbers to represent a vector in n-dimensional vector space. For example, a vector $\vec{v}$ is often represented by v_i, where i runs from 1 to n. Here, v_i denotes the ith component of $\vec{v}$.

In infinite-dimensional vector space, a vector $\vec{v}$ is represented by $v(x)$, where x can be any number between $-\infty$ and ∞. Here, x plays a role, which i plays in finite-dimensional space. In the finite-dimensional case, the dimension of the vector space was n, because i can have only n values, 1 to n. In the infinite-dimensional case, the dimension of the vector space is infinity, because x can have infinite values, any number between the negative infinity and the positive infinity. You may want to think $v(x)$ as the "xth" component of $\vec{v}$. The only difference from the finite-dimensional case is that this x can be any number such as π, $\sqrt{2}$ and -0.334, not just a natural number from 1 to n.

With vectors, we can perform addition, scalar multiplication and in many cases, scalar product.[1] Let's see how they can be done for infinite-dimensional vector space case.

First, let's begin with vector addition. In finite-dimensional vector case, we can add two vectors by adding their components. For example, if we have $\vec{u}+\vec{v}=\vec{w}$, the components of $\vec{w}$ can be obtained by

$$u_i + v_i = w_i. \tag{1}$$

Similarly, in infinite-dimensional vector case, if we have $\vec{u}+\vec{v}=\vec{w}$, $\vec{w}$ can be obtained by

$$u(x) + v(x) = w(x). \tag{2}$$

Second, multiplication by a scalar. $c\vec{u}=\vec{t}$, for a scalar c, corresponds to

$$cu_i = t_i. \tag{3}$$

[1] The definition of vector does not require the existence of scalar product. But, usually, one can endow scalar product to vectors.

Similarly, in infinite-dimensional vector case, $c\vec{u} = \vec{t}$ corresponds to

$$cu(x) = t(x). \tag{4}$$

Finally, scalar product. In finite-dimensional case, we have

$$\vec{u} \cdot \vec{v} = \sum_i u_i v_i. \tag{5}$$

where we see that we have to sum over i. Recall that in infinite-dimensional case, x corresponds to i in finite-dimensional case. Thus, we have to "sum" over x, but "sum" in infinite-dimensional case is integration. Therefore, we have to integrate over x instead. Thus, we obtain

$$\vec{u} \cdot \vec{v} = \int_{-\infty}^{\infty} u(x)v(x)dx \tag{6}$$

Actually, it is very easy to check that the infinite dimensional vector we introduced here satisfies the following eight conditions which any vector must satisfy by definition.

- 1) Vector addition must be associative. This condition is satisfied for $u(x)$, $v(x)$, and $w(x)$ because $u(x) + (v(x) + w(x)) = (u(x) + v(x)) + w(x)$.
- 2) Vector addition must be commutative. This condition is satisfied because $u(x) + v(x) = v(x) + u(x)$.
- 3) Vector addition must have an identity element. This condition is satisfied because $u(x) + 0 = u(x)$, where 0 denotes a constant function 0 for all x.
- 4) Vector addition must have inverse elements. This condition is satisfied because $u(x) + -u(x) = 0$, where $-u(x)$ is the additive inverse of $u(x)$.
- 5) Distributivity must hold for scalar multiplication over vector addition. This condition is satisfied because $a(u(x)+v(x)) = au(x)+av(x)$.
- 6) Distributivity must hold for scalar multiplication over field addition. This condition is satisfied because $(a + b)u(x) = au(x) + bu(x)$.
- 7) Scalar multiplication must be compatible with multiplication in the field of scalars. This condition is satisfied because $a(bu(x)) = (ab)u(x)$.
- 8) Scalar multiplication must have an identity element. This condition is satisfied because $1u(x) = u(x)$.

Now, let's slightly change our focus. If we have vectors, we can consider a function from vector to vector. If such a function is linear, we call it a linear operator or a matrix. As in finite-dimensional case, we can consider such linear operators in infinite-dimensional case.

Surprisingly, we will see that multiplication by x and differentiation with respect to x are both linear operators (or matrices).

Let me explain. By definition, a linear operator (or matrix) L must satisfy the following two conditions:

$$L(x+y) = L(x) + L(y)$$

$$L(ax) = aL(x)$$

Multiplication by x satisfies the above conditions because $x(u(x)+v(x)) = xu(x)+xv(x)$ and $x(au(x)) = axu(x)$. Differentiation with respect to x satisfies the above conditions because

$$\frac{\partial(u(x)+v(x))}{\partial x} = \frac{\partial u(x)}{\partial x} + \frac{\partial v(x)}{\partial x}$$

and

$$\frac{\partial(au(x))}{\partial x} = a\frac{\partial u(x)}{\partial x}$$

Therefore, multiplication by x and differentiation with respect to x are linear operators (or matrices).

In case of the finite-dimensional vector space, we learned that matrix multiplication is not generally commutative, i.e., it doesn't satisfy $AB = BA$. In other words, the following two expressions

$$\vec{v} \xrightarrow{A} A\vec{v} \xrightarrow{B} BA\vec{v} \tag{7}$$

$$\vec{v} \xrightarrow{B} B\vec{v} \xrightarrow{A} AB\vec{v} \tag{8}$$

are not the same in general.

Actually, it's easy to check that the linear operators, multiplication by x and differentiation with respect to x do not commute. Let's see this.

$$v(x) \xrightarrow{x} xv(x) \xrightarrow{\frac{\partial}{\partial x}} \frac{\partial(xv(x))}{\partial x} \tag{9}$$

$$v(x) \xrightarrow{\frac{\partial}{\partial x}} \frac{\partial v(x)}{\partial x} \xrightarrow{x} x\frac{\partial v(x)}{\partial x} \tag{10}$$

Problem 1. Let's call the linear operator that multiplies by x, A and the linear operator that differentiates with respect to x, B. Then, show that $AB - BA = -I$, where I is the identity operator.

Finally, we will introduce a notation. In this article, we expressed vectors only by the components. But of course, we can denote vectors by the components with basis. For example, if $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n$ is the basis, we have

$$\vec{v} = \sum_{i=1}^{n} v_i \vec{e}_i \tag{11}$$

Here, we see that i is the label that denotes each basis $\vec{e}_i$ and v_is are the components for $\vec{v}$.

Similarly, an infinite-dimensional vector $\vec{u}$ can be regarded as the following vector:

$$\vec{u} = \int_{-\infty}^{\infty} dx\, u(x)|x\rangle \tag{12}$$

Here we see that x is the label that denotes each basis $|x\rangle$ and $u(x)$ is the "component" for $\vec{u}$. Also, as x can be any value between $-\infty$ and ∞, we have integration instead of the sum as in (11). We see indeed that the vector space here is infinite-dimensional, as there are infinitely many values for x which label the basis. We will talk more about the notation $|x\rangle$ in our later article on Dirac's bra-ket notation.

Summary

- The dimension of a vector space can be infinite-dimensional.
- As much as u_i is the ith component for $\vec{u}$, in infinte-dimensional case, $v(x)$ can be regarded as the "xth" component for $\vec{v}$. x can be any value between $-\infty$ and ∞, thus the vector space is infinite-dimensional.
- $\vec{u} + \vec{v} = \vec{w}$ means $u(x) + v(x) = w(x)$.
- As much as $\vec{u} \cdot \vec{v} = \sum_i u_i v_i$, we have

$$\vec{u} \cdot \vec{v} = \int_{-\infty}^{\infty} u(x)v(x)dx$$

- Multiplication by x and differentiation with respect to x are linear operators (i.e. matrices) because they satisfy the requirement of linearity.