## Integration by substitution

Suppose you want to calculate following:

$$
\begin{equation*}
\int 8 x^{3} e^{x^{4}} d x \tag{1}
\end{equation*}
$$

At first glance, it may seem hard to calculate. However, there is a good trick to solve such a problem. First, we substitute

$$
\begin{equation*}
t=x^{4} \tag{2}
\end{equation*}
$$

Then, we have:

$$
\begin{equation*}
\frac{d t}{d x}=4 x^{3} \rightarrow d t=4 x^{3} d x \tag{3}
\end{equation*}
$$

Given this, (1) can be re-expressed as:

$$
\begin{equation*}
\int 2 e^{x^{4}}\left(4 x^{3} d x\right)=\int 2 e^{t} d t=2 e^{t}+C=2 e^{x^{4}}+C \tag{4}
\end{equation*}
$$

This trick is called "integration by substitution," as you substitute the integration variable by something else.

Problem 1. Calculate the following. (Hint ${ }^{1}$ )

$$
\begin{equation*}
\int \frac{(\ln x)^{4}}{x} d x, \quad \int x^{2} \cos \left(x^{3}\right) d x \tag{5}
\end{equation*}
$$

Problem 2. Show the following. This trick due to Feynman is useful in quantum field theory later. (Hint ${ }^{2}$ )

$$
\begin{equation*}
\frac{1}{A B}=\int_{0}^{1} \frac{d x}{(A(1-x)+B x)^{2}} \tag{6}
\end{equation*}
$$

Problem 3. Calculate the following. (Hint ${ }^{3}$ )

$$
\begin{equation*}
\int \frac{1}{\sqrt{1-x^{2}}} d x, \quad \int \frac{1}{1+x^{2}} d x \tag{7}
\end{equation*}
$$

Problem 4. Calculate the following and explain why the answer must coincide with the area of a semi-circle with radius 1 . (Hint ${ }^{4}$ )

$$
\begin{equation*}
\int_{-1}^{1} \sqrt{1-x^{2}} d x \tag{8}
\end{equation*}
$$

[^0]Problem 5. Calculate the following, assuming that $m>2$. (Hint ${ }^{5}$ )

$$
\begin{equation*}
\int \frac{l^{3}}{\left(l^{2}+\Delta\right)^{m}} d l \tag{9}
\end{equation*}
$$

Problem 6. A Korean child prodigy Ung-yong Kim solved the following problem when he was four years old appearing on a Japanese TV-show in 1967. Let's see whether you can solve it as well! (Hint ${ }^{6}$ )

$$
\begin{equation*}
\int \frac{2 x}{\sqrt{4-3 x^{2}}} d x=? \tag{10}
\end{equation*}
$$

Problem 7. Calculate the following. This problem is useful to solve in our later article "Another example of differential equation." (Hint ${ }^{7}$ )

$$
\begin{equation*}
\int \frac{1}{\left(1+x^{2}\right)^{3 / 2}} d x \tag{11}
\end{equation*}
$$

Problem 8. Calculate the following. This problem is useful to calculate the ground state energy of Helium atom later:

$$
\begin{equation*}
\int \frac{d \theta}{\sqrt{r_{1}^{2}+r_{2}^{2}-2 r_{1} r_{2} \cos \theta}} \tag{12}
\end{equation*}
$$

Problem 9. The following is known.

$$
\begin{equation*}
\int_{-\infty}^{\infty} e^{-x^{2}} d x=\sqrt{\pi} \tag{13}
\end{equation*}
$$

Given this, evaluate the following for $a>0$ :

$$
\begin{equation*}
\int_{-\infty}^{\infty} e^{-a x^{2}} d x \tag{14}
\end{equation*}
$$

Integration of this form is called "Gaussian integral" and often used in statistics. (We will see this in our later article "Gaussian distribution.") Notice that it is expressed in terms of $\pi$. In "the Unreasonable Effectiveness of Mathematics in the Natural Sciences," late Nobel laureate Eugene Wigner introduces an interesting anecdote.

THERE IS A story about two friends, who were classmates in high school, talking about their jobs. One of them became a statistician and was working on population trends. He showed a reprint to his former classmate. The reprint started, as usual, with the Gaussian distribution and the statistician explained to his former classmate the meaning of the symbols for the actual population, for the average population, and so on. His classmate was a bit incredulous and was not quite sure whether the statistician was pulling his leg. "How can you know that?" was his query. "And what is this symbol here?" "Oh," said the statistician, "this is pi." "What is that?" "The ratio of the circumference of the circle to its diameter." "Well, now you are pushing your joke too far," said the classmate, "surely the population has nothing to do with the circumference of the circle."

[^1]Now you can easily imagine that the population can have something to do with the circumference of the circle. We will use Gaussian integral in our later article on Feynman diagram and prove (13) in "Polar coordinate, the area of a circle and Gaussian integral."

## Summary

- In some cases, you can perform integration by substitution.


[^0]:    ${ }^{1}$ Set $t=\ln x$ for the first one. Set $t=x^{3}$ for the second one.
    ${ }^{2}$ Set $t=A(1-x)+B x$.
    ${ }^{3}$ For the first one, substitute by $x=\sin \theta$. For the second one, substitute by $x=\tan \theta$.
    ${ }^{4}$ Substitute by $x=\sin \theta$. For the second part of problem, draw the graph $y=\sqrt{1-x^{2}}$.

[^1]:    ${ }^{5}$ Substitute by $t=l^{2}+\Delta$
    ${ }^{6}$ Set $t=4-3 x^{2}$
    ${ }^{7}$ Substitute by $x=\tan \theta$. After the substitution and the integration, Problem 3 in our earlier article "What are trigonometric functions?" is helpful.

