Intermediate value theorem

Intermediate value theorem is a theorem about continuous functions. Let's say $f(x)$ is a function continuous at the interval $a \leq x \leq b$. Then, the theorem says that for any u between the values $f(a)$ and $f(b)$ (i.e. if $f(a) \leq f(b)$, then $f(a) \leq u \leq f(b)$, if $f(a) \geq f(b)$, then $f(a) \geq u \geq f(b)$) there is at least one $a \leq c \leq b$ that satisfies $u = f(c)$. This sounds somewhat abstract without a graph, so see Fig. 1. $f(x)$ is continuous for the interval $a \leq x \leq b$. In other words, the graph is connected (i.e. not "broken") in this interval. Therefore, there is no way that $f(x)$ can go from $(a, f(a))$ to $(b, f(b))$ without crossing the line $y = u$, if u is between $f(a)$ and $f(b)$. You indeed see that there is a $a \leq c \leq b$ that satisfies $u = f(c)$.

In Fig. 2, you see an example of a discontinuous function $y = \tan x$, and how intermediate value theorem doesn't apply to a discontinuous function. Here, we are considering the interval $\pi/4 \leq x \leq 3\pi/4$. The graph is discontinuous at $x = \pi/2$. Therefore, even though the value $1/2$ is between $\tan \dfrac{\pi}{4} = 1$ and $\tan \dfrac{3\pi}{4} = -1$, there is no need for the existence of $\pi/4 \leq x \leq 3\pi/4$ that satisfies $\tan x = \dfrac{1}{2}$. In the graph, you actually see that $\tan x = \dfrac{1}{2}$ doesn't have the solution for the interval $\pi/4 \leq x \leq 3\pi/4$.

We will not bother to prove this theorem, because it is intuitively clear. If you are interested in the proof, look up Wikipedia.

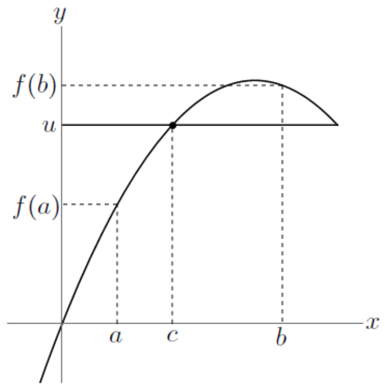

Figure 1: $y = f(x), y = u$

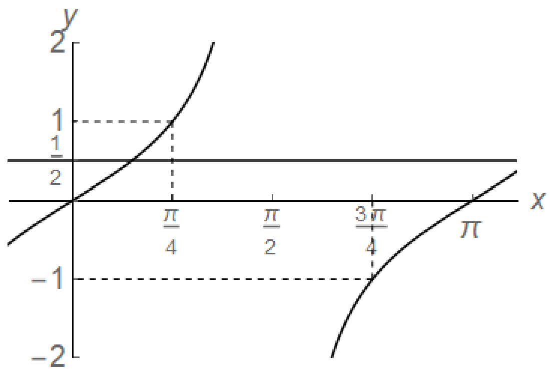

Figure 2: $y = \tan x, y = \dfrac{1}{2}$

Summary

- Let's say $f(x)$ is a function continuous at the interval $a \leq x \leq b$. Then, the theorem says that for any u between the values $f(a)$ and $f(b)$ there is at least one $a \leq c \leq b$ that satisfies $u = f(c)$.