Matrix inverses

In this article, I will explain what inverse matrix is. To this end, I begin by defining a matrix. An "$i \times j$ matrix" is a linear map that accepts j numbers and spits out i numbers. Given the particular form of such a matrix, an interesting question one can ask is the following:

"If we know the output of numbers from the linear map or matrix, would it be possible to deduce what the input of numbers for the linear map or matrix was?"

Let's try to solve this problem when the matrix is a 2×2 matrix. A 2×2 matrix A can be represented as following:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{1}$$

As this is a 2×2 matrix, if we enter two numbers "x_1, x_2" we will get two numbers "y_1, y_2" as follows.

$$AX = Y = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{2}$$

Here we have denoted the two numbers we entered as matrix X and the output of the two numbers as matrix Y.

Explicitly, the above equation means the following:

$$a_{11}x_1 + a_{12}x_2 = y_1 \tag{3}$$
$$a_{21}x_1 + a_{22}x_2 = y_2 \tag{4}$$

So, the problem concerned is obtaining x_1 and x_2, when we have y_1 and y_2. This we can do, as it is a simple arithmetic to obtain x_1 and x_2 in terms of y_1 and y_2.

We obtain:

$$x_1 = (a_{22}y_1 - a_{12}y_2)/(\det A) \tag{5}$$
$$x_2 = (-a_{21}y_1 + a_{11}y_2)/(\det A) \tag{6}$$

where we defined $\det A = a_{11}a_{22} - a_{21}a_{12}$ for future convenience. It will be called the "determinant" of matrix A.

In matrix notation we can rewrite the above formula as follows

$$X = A^{-1}Y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{7}$$

where we have defined:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{8}$$

This is called the inverse matrix of A. In order to explain why it is denoted as A^{-1}, let a, x, y be three numbers which satisfy the following relation:

$$ax = y \tag{9}$$

Then, what would be x in terms of a and y?

This would be simply

$$x = a^{-1}y \tag{10}$$

where $aa^{-1} = 1$. Let's check this. Plugging (10) to (9), we get

$$a(a^{-1}y) = y \tag{11}$$

Thus, a^{-1} should satisfy $aa^{-1} = 1$.

Hence, we can understand why we use such a notation for the inverse of matrix.

Now, let's think about some properties of the inverse matrix. If we have

$$AX = Y \tag{12}$$

the inverse matrix is defined by

$$A^{-1}Y = X \tag{13}$$

Plugging this equation to (12), we get

$$AA^{-1}Y = Y \tag{14}$$

Since this is true for any Y, we conclude that

$$AA^{-1} = I \tag{15}$$

where I is the identity matrix. In other words, AA^{-1} spits out the same numbers as entered, hence it must be the identity matrix. Also observe the following. If we multiply by A on both-hand side of the above equation, we get:

$$AA^{-1}A = IA = A \tag{16}$$

$$A(A^{-1}A) = A \tag{17}$$

As we have $AI = A$, we conclude:

$$A^{-1}A = I \tag{18}$$

It is a simple exercise that the inverse matrix for an arbitrary 2×2 matrix indeed satisfies the above two conditions. That is:

$$AA^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = I \tag{19}$$

$$A^{-1}A = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = I \tag{20}$$

Check yourself!

Square matrix is a matrix that has equal number of rows and columns. i.e. an $i \times j$ matrix where $i = j$. The inverse matrix can be defined for any arbitrary square matrix A unless $\det A = 0$. Even though we haven't discussed the determinant of a square matrix bigger than 2×2 in this article, the determinant can be defined for any arbitrary square matrix.

For non-square matrix, it is a bit subtle. An inverse of $i \times j$ matrix means obtaining j unknowns given i equations. If j is bigger than i, we cannot completely determine the unknowns. So, there is no unique inverse. If i is bigger than j, as there are more equations than unknowns the solutions don't exist for generic cases. So, generally, there is no inverse either.

Also, note that there is no inverse, if $\det A$ is zero, as one cannot divide by zero in (8). In such cases, there are either infinite solutions for the equation or no solution at all. For example, the determinant of the following matrix is zero,

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \tag{21}$$

as $2 \times 6 - 3 \times 4 = 0$

Then, the corresponding equations for this matrix are:

$$2x_1 + 3x_2 = y_1 \tag{22}$$

$$4x_1 + 6x_2 = y_2 \tag{23}$$

Therefore, if $y_2 = 2y_1$, there are infinite solutions. Otherwise, there are no solutions at all.

Finally, we want to remark that the inverse matrix of A doesn't exist, if there is non-zero X that satisfies

$$AX = 0 \tag{24}$$

From (12) and (13), we see that if A^{-1} existed, it must satisfy

$$A^{-1}0 = X \tag{25}$$

However, the left-hand side is 0, because you get 0 if you multiply any matrix by 0. Therefore, the right-hand side must be 0, but we said that X is a non-zero vector. So, the equality cannot hold. The only conclusion we can draw is that A^{-1} doesn't exist. You can actually see it more easily, if you consider the analogy with numbers. Let $ax = 0$ for a non-zero x. Then, a must be 0 and there is no a^{-1}, as there is no inverse for 0. In case of a number, a has to be strictly 0 in order to not have an inverse. In case of a matrix, A doesn't necessarily have to be 0, and having its determinant equal to 0 is sufficient to not have an inverse.

If a matrix has an inverse, we call such a matrix invertible. We will later see that a matrix is invertible, if and only if the determinant is non-zero. We will also see that $\det A$ is zero (i.e. A is not invertible), if and only if there is non-zero vector X that satisfies $AX = 0$. We will also see how we can calculate determinant of any square matrices, not just 2×2.

Problem 1. Show the following:

$$(AB)^{-1} = B^{-1}A^{-1} \tag{26}$$

Problem 2. In our earlier article "Matrices and Linear Algebra," you obtained the matrix for the anti-clockwise rotation by angle θ. Let's call this matrix $R(\theta)$. Now, you will obtain its inverse by using two methods. First, use (8). Second, explain in words why the inverse of $R(\theta)$ must be $R(-\theta)$, and check that $R(-\theta)$ agrees with the inverse of $R(\theta)$ obtained by the first method.

Problem 3. In our earlier article "Matrices and Linear Algebra," you realized that the multiplication of two $n \times n$ diagonal matrices is a $n \times n$ diagonal matrix. Using this fact, obtain the inverse for the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \tag{27}$$

Summary

- The determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $ad - bc$.
- A^{-1}, the inverse matrix of A is a matrix that always satisfies $A^{-1}Y = X$, if $AX = Y$.
- $A^{-1}A = AA^{-1} = I$.