Levi-Civita symbol

Levi-Civita symbol in 2-dimensions, denoted ϵ_{ij}, where i, j can take values from 1 to 2 (hence 2-dimensions), is defined by the following rules:

$$\epsilon_{12} = 1, \qquad \epsilon_{ji} = -\epsilon_{ij} \tag{1}$$

In other words, the second rule says that you get an extra negative sign if you exchange the indices. These rules implie following:

$$\epsilon_{21} = -\epsilon_{12} = -1 \tag{2}$$

$$\epsilon_{11} = -\epsilon_{11} \tag{3}$$

$$\epsilon_{11} = 0 \tag{4}$$

Similarly, we have $\epsilon_{22} = 0$.

Levi-Civita symbol in 3d is defined by similar rules. First, we have:

$$\epsilon_{123} = 1 \tag{5}$$

Second, if we exchange any two of the indices, you get an extra negative sign as follows:

$$\epsilon_{jik} = -\epsilon_{ijk} \tag{6}$$

$$\epsilon_{ikj} = -\epsilon_{ijk} \tag{7}$$

$$\epsilon_{kji} = -\epsilon_{ijk} \tag{8}$$

Now, one can check that this implies (**Problem 1.**):

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \tag{9}$$

$$\epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1 \tag{10}$$

$$\epsilon_{ijk} = 0, \quad \text{if} \quad i = j \quad \text{or} \quad j = k \quad \text{or} \quad k = i \tag{11}$$

By now, the readers may have guessed what the general form for Levi-Civita symbol in higher dimensions. They are defined by following rules:

$$\epsilon_{123\cdots n} = 1 \tag{12}$$

$$\epsilon_{\cdots i \cdots j \cdots} = -\epsilon_{\cdots j \cdots i \cdots} \tag{13}$$

where, as before, the second rule implies that the Levi-Civita symbol gets an extra negative sign when any two of the indices are exchanged. This also implies that if any of the indices are repeated, Levi-Civita symbol vanishes. We have already seen this in (11).

At this point, let us introduce important terminologies. An even permutation is a permutation one can obtain by performing two-element swaps even numbers. An odd permutation is a permutation one can obtain by performing two-element swaps odd numbers. For example, if the original set is (1,2,3), from (9) we see that (1,2,3), (2,3,1) and (3,1,2) are even permutations as performing two-element swaps even times makes the Levi-Civita symbol multiplied by (-1) even times, which make the Levi-Civita symbol 1. Similarly, from (10) we see that (2,1,3), (1,3,2) and (3,2,1) are odd permutations. A permutation is either even permutation or odd permutation, but never both. Otherwise, if we calculate the Levi-Civita symbol for the permutation corresponding to it, it would be 1 and -1 *at the same time*, which is impossible.

Sometimes, it's useful to use Levi-Civita symbol for upper indices. For example,

$$\epsilon^{123\cdots n} = 1, \qquad \epsilon^{\cdots i \cdots j \cdots} = -\epsilon^{\cdots j \cdots i \cdots} \tag{14}$$

However, in general relativity in 4 dimensional spacetime, indices run from 0 to 3, instead of 1 to 4. In that case, it turns out that it's more convenient to define $\epsilon^{0123} = -1$ while $\epsilon_{0123} = 1$.

Now, let us explain some useful notations which we will use soon. First, we introduce symmetrizer "()"as follows:

$$T_{(ab)} = T_{ab} + T_{ba} \tag{15}$$

Some books use the following different convention for symmetrizer.

$$T_{(ab)} = \frac{1}{2}T_{ab} + \frac{1}{2}T_{ba} \tag{16}$$

Second, as follows, we introduce antisymmetrizer "[]" which is much more frequently used than symmetrizer.

$$T_{[ab]} = T_{ab} - T_{ba} \tag{17}$$

Some books use the following different convention for antisymmetrizer.

$$T_{[ab]} = \frac{1}{2}T_{ab} - \frac{1}{2}T_{ba} \tag{18}$$

If we use this convention, if S_{ab} is an antisymmetric rank 2-tensor, we have:

$$S_{[ab]} = S_{ab} \tag{19}$$

Antisymmetrizer can be generalized to the case in which there are more than two indices. For example,

$$T_{[abc]} = T_{abc} + T_{bca} + T_{cab} - T_{acb} - T_{bac} - T_{cba} \tag{20}$$

We see here that the sign in front of Ts are 1 for even permutations while they are -1 for odd permutations. Some books use the following different convention for antisymmetrizer.

$$T_{[abc]} = \frac{1}{3!}(T_{abc} + T_{bca} + T_{cab} - T_{acb} - T_{bac} - T_{cba}) \tag{21}$$

If we use this convention, if S_{abc} is an antisymmetric rank 3-tensor, we have:

$$S_{[abc]} = S_{abc} \tag{22}$$

If we didn't use this convention, there would be extra 3! on the right hand-side of the above equation. In other words, some physicists use this convention to get rid of this extra factor.

Given this, we introduce how a product of two Levi-Civita symbols such as $\epsilon_{ij}\epsilon^{kl}$ can be expressed in terms of Kronecker delta symbol. Let's start with the easiest case, namely when there are two indices. First, we have:

$$\epsilon_{12}\epsilon^{12} = \delta^1_1\delta^2_2 \tag{23}$$

$$\epsilon_{21}\epsilon^{21} = \delta^2_2\delta^1_1 \tag{24}$$

This suggests something like

$$\epsilon_{ij}\epsilon^{kl} = \delta^k_i\delta^l_j \tag{25}$$

However, this is wrong. The left-hand side is antisymmetric with respect to the indices i and j while the right-hand side is not. Therefore, we have to antisymmetrize the right-hand side. Therefore, we have:

$$\epsilon_{ij}\epsilon^{kl} = \delta^k_{[i}\delta^l_{j]} = \delta^k_i\delta^l_j - \delta^k_j\delta^l_i \tag{26}$$

You can easily check this is correct. Similarly,

$$\epsilon_{ijk}\epsilon^{lmn} = \delta^l_{[i}\delta^m_j\delta^n_{k]} \tag{27}$$

Let us conclude this article with a comment. In my paper "Black hole entropy predictions without Immirzi parameter and Hawking radiation of single-partition black hole" with Brian Kong, I asserted that Rovelli's formula for "area operator" was wrong, because he used Levi-Civita symbol for the place in which he should have used Levi-Civita tensor. "Area operator" gives you the value for the area spectrum. To learn what the area spectrum is, please read my article "Discrete area spectrum and black hole entropy I." To learn what Levi-Civita tensor is, please read my article "A short introduction to general relativity."

Problem 2. What is ϵ_{4123}? What is ϵ_{2314}?

Problem 3. What is $\epsilon_{abc}\epsilon^{abc}$?

Problem 4. Express $\epsilon_{abc}\epsilon^{cde}$ in terms of Kronecker delta symbols.

Problem 5. Express $\epsilon_{abcd}\epsilon^{efgh}$ using in terms of Kronecker delta symbols and an antisymmetrizer, where the indices run from 0 to 3 instead of 1 to 4.

Summary

- Levi-Civita symbol is defined by

$$\epsilon_{123\cdots n} = 1$$

$$\epsilon_{\cdots i \cdots j \cdots} = -\epsilon_{\cdots j \cdots i \cdots}$$

- The Levi-Civita symbol is zero if any of the indices are repeated.
- Symmetrizers are denoted by "()."
- Anti-symmetrizers are denoted by "[]."
- A product of two Levi-Civita symbols can be expressed using Kronecker delta symbol and appropriate anti-symmetrizers.