L'Hôpital's rule

In our earlier article "What is a limit?" we have introduced the concept of limit. In the articles following it, we have applied the concept of limit to introduce the concept of derivatives, and obtained explicit formulas for derivatives.

In this article, we will present "l'Hôpital's rule," a handy rule to obtain limit using differentiation. Of course, the concept of limit doesn't presuppose the concept of differentiation. On the contrary, the concept of differentiation presupposes the concept of limit. Therefore, it would be ludicrous to use l'Hôpital's rule to calculate limits that are needed to derive the rules for differentiation. It would be circular reasoning. Nevertheless, as long as we take the rules for differentiation are given, l'Hôpital's rule is so simple that you can solve most, if not all, problems on limit if they appear on exams.

Given this, let's begin our discussion. Suppose you are asked to obtain the value for following:

$$\lim_{x\to c}\frac{f(x)}{g(x)} \tag{1}$$

In most cases, the answer will be simply $f(c)/g(c)$. The only cases that we have to be careful are when $f(c)$ and $g(c)$ are zero or infinity. Strictly speaking, we only need to be careful when they are both zero or both infinity, as when $f(c) = 0$ and $g(c) = \infty$, the limit of $f(x)/g(x)$ is zero and when $f(c) = \infty$ and $g(c) = 0$, the limit of $f(x)/g(x)$ is either infinity or negative infinity or doesn't exist depending the cases.

Now, let me state l'Hôpital's rule. If both $f(c)$ and $g(c)$ are zero or if the limits of $f(x)$ and $g(x)$ as x approaches c are infinity or negative infinity, we have:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)} \tag{2}$$

The proof for the first case (i.e. $f(c) = g(c) = 0$) is easy. We have:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f(x)-0}{g(x)-0} = \lim_{x\to c}\frac{f(x)-f(c)}{g(x)-g(c)} = \lim_{x\to c}\frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} = \lim_{x\to c}\frac{f'(x)}{g'(x)} \tag{3}$$

The proof for the second case (i.e. $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty$ is a bit more difficult. First, we have to introduce y that satisfies $x < y < c$. Then, we have:

$$\begin{aligned}\lim_{x\to c}\frac{f'(x)}{g'(x)} &= \lim_{x\to c}\left(\lim_{y\to c}\frac{f(y)-f(x)}{g(y)-g(x)}\right) = \lim_{x\to c}\left(\lim_{y\to c}\frac{\frac{f(y)}{g(y)}-\frac{f(x)}{g(y)}}{1-\frac{g(x)}{g(y)}}\right)\\ &= \lim_{x\to c}\left(\lim_{y\to c}\frac{\frac{f(y)}{g(y)}-0}{1-0}\right) = \lim_{y\to c}\frac{f(y)}{g(y)}\end{aligned} \tag{4}$$

where from the first line to the second line we have used the fact that $g(y)$ approaches the infinity much faster than $f(x)$ or $g(x)$ as y is closer to c than x is and as the limit that y approaches c is first taken before the limit that x approaches c. In conclusion, the above formula is exactly (2).

Let us show you an example.

$$\begin{aligned}\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x-\sin x} &= \lim_{x\to 0}\frac{2\cos x-2\cos 2x}{1-\cos x} = \lim_{x\to 0}\frac{-2\sin x+4\sin(2x)}{\sin x} \\ &= \lim_{x\to 0}\frac{-2\cos x+8\cos(2x)}{\cos x} = \frac{-2+8}{1} = 6 \end{aligned} \tag{5}$$

Problem 1.

$$\lim_{x\to 0}\frac{x\sin x}{1-\cos x} = ? \tag{6}$$

Problem 2. (Hint[1])

$$\lim_{x\to 0} x\ln x = ? \tag{7}$$

Problem 3. Use the result of Problem 2 to calculate the following. (Hint[2])

$$\lim_{x\to 0} x^x = ? \tag{8}$$

Summary

- If both $f(c)$ and $g(c)$ are zero or if the limits of $f(x)$ and $g(x)$ as x approaches c are infinity or negative infinity, l'Hôpital's rule states that

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

[1] Re-express the formula as $\lim_{x\to 0}\frac{\ln x}{1/x}$, then use L'Hôpital's rule.

[2] Use $e^{x\ln x} = x^x$.