Lie group

So far, we have only seen groups whose order are finite, or infinite but countable. (For those of you who don't know what "countable" means, I remark that there are two kinds of infinity: countable and uncountable. As an example, the number of integer is infinite, but it's countable, whereas the number of real number between 1 and 2 is uncountable. Roughly speaking, the number of real number between 1 and 2 is so huge that it is uncountable, whereas the number of integer is not as huge, which makes it countable. In our case, we have seen that integer with addition as group action forms a group. This group has infinite order, but it is nevertheless countable.)

However, there are groups that have uncountably infinite order. They are called "Lie group." (Lie pronounced as "Lee.") Actually, Lie group is a group that is also manifold.

Let me give you an example of Lie group. Arbitrary rotation around fixed axis forms a group. For example, 3.4° clockwise rotation followed by 150° clockwise rotation results in 153.4° clockwise rotation. Likewise, the inverse of 11.1° clockwise rotation is 11.1° anti-clockwise rotation, which is the same thing as 348.9° clockwise rotation.

It is easy to see that each element in this group corresponds to any number between 0 and 360°, or if you are more familiar with radians, any number between 0 and 2π. It is also easy to see that this group is the manifold S^1, the circle, which we talked about in our earlier article on manifold. (In other words, each element in this group corresponds one to one with each point in the manifold S^1.)

Another representation of this group is regarding $e^{i\theta}$ as each element where θ is a real number, which denotes the rotation angle in radians, and the group action as multiplication.

For example, that 270° clockwise rotation followed by 90° clockwise rotation is the same thing as no rotation at all, can be represented as

$$e^{3\pi i/2}e^{\pi i/2} = e^{0i} = 1 \tag{1}$$

I would have loved to give you other examples of Lie group, but this would be postponed until you learn about matrices. At this point, I just want to comment that Lie group is completely classified.

Anyhow, Lie group is important in quantum mechanics. When Sophus Lie, a Norwegian mathematician discovered groups that are now named after him, he perhaps didn't know that his work would be one day important in physics. Many branches of mathematics are just like that. They were created without mind in application in physics, but they are now so useful in physics.

Summary

- Lie group is a group that is also manifold. It has infinite order.
- Rotation through a fixed axis forms a Lie group, which is the same thing as S^1.