Light as electromagnetic waves

What are Maxwell's equations in vacuum? We know that there is no electric charge in vacuum. Therefore, $\rho = 0$. This implies:

$$\nabla \cdot \vec{E} = 0 \tag{1}$$

$$\nabla \cdot \vec{B} = 0 \tag{2}$$

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t} \tag{3}$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{4}$$

Now, for any vector $\vec{U}$, the following identity is known:

$$\nabla \times (\nabla \times \vec{U}) = \nabla(\nabla \cdot \vec{U}) - \nabla^2 \vec{U} \tag{5}$$

where ∇^2 is called "Laplacian" and defined by $\nabla^2 f = \nabla \cdot (\nabla f)$. You can prove this identity by using the result of Problem 4 in our earlier article "Levi-Civita symbol," as you did so when proving the identity in Problem 2 in our earlier article "The cross product."

Given this, let's apply curl to (3). We have:

$$\begin{aligned}
\nabla \times (\nabla \times \vec{E}) &= \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \\
\nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right) &= -\nabla^2 \vec{E} \\
\frac{\partial}{\partial t} \left(\nabla \times \vec{B} \right) &= \nabla^2 \vec{E} \\
\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} &= \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2}
\end{aligned} \tag{6}$$

Similarly, by applying curl to (4) and taking similar steps, we obtain:

$$\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = \frac{\partial^2 \vec{B}}{\partial x^2} + \frac{\partial^2 \vec{B}}{\partial y^2} + \frac{\partial^2 \vec{B}}{\partial z^2} \tag{7}$$

However, if you remember our earlier article "Partial differential equation" (6) and (7) are exactly the equations for waves with velocity

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \tag{8}$$

Therefore, (6) and (7) are equations for electromagnetic waves! Maxwell obtained this result, and it coincided with the speed of light known at the time. Therefore, he showed that it was quite likely that electromagnetic waves are light. Later, this was experimentally confirmed by Hertz in the late 19th century.

Now, let's consider a solution to (6). We have (**Problem 1.** Check this!):

$$\vec{E} = \sum_{\vec{k}} \vec{E}_{\vec{k}} e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{9}$$

where we have $\omega = |\vec{k}|c$ where $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$.[1]

Now, let's consider a monochromatic light that travel in a certain direction. (Monochromatic means that we have a single $|\vec{k}|$.) Then, the above equation can be re-written as:

$$\vec{E} = \vec{E}_{\vec{k}} e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{10}$$

Plugging this into (1), we get:

$$\vec{k} \cdot \vec{E}_{\vec{k}} = 0 \tag{11}$$

So, we have an extra condition that $\vec{E}_{\vec{k}}$ must satisfy. Let us explain its significance. At first glance, an electromagnetic wave seems to have three degrees of freedom (i.e. independent components) as it has three components (i.e. x, y, z components) as a vector. However, the extra condition removes one of them, which makes the degree of freedom of an electromagnetic wave two. This we say light has two polarizations. The same conclusion can be obtained by considering the magnetic field which is given by

$$\vec{B} = \sum_{\vec{k}} \vec{B}_{\vec{k}} e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{12}$$

For a monochromatic right, we have

$$\vec{B} = \vec{B}_{\vec{k}} e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{13}$$

From (2), we get:

$$\vec{k} \cdot \vec{B}_{\vec{k}} = 0 \tag{14}$$

So, $\vec{B}_{\vec{k}}$ has two degrees of freedom. At this point, a confused reader may think that light has four degrees of freedom as its electric field has two while its magnetic field has two. But this is not correct as magnetic field is determined once electric field is determined and vice versa by (3) and (4).

Problem 2. Let $\vec{k} = (0, 0, k_z)$. Which of the components (i.e. x, y, z components) of $\vec{E}_{\vec{k}}$ must be zero?

Problem 3. In such a case, express $\vec{B}_{\vec{k}} = (B_x, B_y, B_z)$ in terms of $\vec{E}_{\vec{k}} = (E_x, E_y, E_z)$ components by components. (Hint[2])

The polarization of light can be used to play a 3D movie in a theater. Let me explain. When you watch a 3D movie, you wear special glasses. The left lens of the glasses allows

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[1] This exponential solution is actually equivalent to the more familiar sine and cosine solutions. See "Forced harmonic oscillator" for the reason why.

[2] From (3), (10) and (13), show $\vec{k} \times \vec{E} = \omega \vec{B}$.

say, only the light of which the electric field oscillate vertically, to penetrate, and the right lens of the glasses allows say, only the light of which the electric field oscillate horizontally, to penetrate. Then, if you project on the screen the image which the left eye would see, in light whose electric field is vertically polarized, and the image which the right eye would see, in light whose electric field is horizontally polarized, you will see the image on the screen as if you see the real image with two eyes in daily lives.

In this article, we have seen that the oscillation direction is perpendicular to the propagation direction for the electromagnetic waves. Thus, light is indeed a transverse wave, as mentioned in "Transverse waves and longitudinal waves."

Problem 4. Show that the form of Maxwell's equations in vacuum is preserved under

$$\vec{E} \to \vec{B}, \qquad \vec{B} \to -\frac{1}{c^2}\vec{E} \tag{15}$$

This is known as "electromagnetic duality." We will talk more about it in "Electromagnetic duality"

Summary

- Light is an electromagnetic wave. Its speed is given by $c = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}}$.
- Light is a transverse wave, as its direction of propagation is perpendicular to its electric component and its magnetic component. Its electric components and its magnetic components are perpendicular to each other.
- There are two polarizations of light.